メタ階層ダイナミクスユニットセミナー
Meta-hierarchy Dynamics Unit Seminar
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複雑な階層性を持つプラズマ現象の理論解明
Modeling Plasma Phenomena with Complex Hierarchies

① 曲率を持つ磁場における一般化長谷川三間方程式の開発

A Generalized Hasegawa-Mima Equation for Drift Wave Turbulence in Curved Magnetic Fields
[1] N. Sato and M. Yamada 2022 J. Plasma Phys. 88 3

② 三次元閉じ込め磁場の存在問題に関する数理解析

Existence of Three-Dimensional Magnetohydrostatic Equilibria
[2] N. Sato and M. Yamada 2023 J. Math. Phys. 64, 081505

核融合科学研究所 メタ階層ダイナミクスユニット 佐藤直木
Naoki Sato, Meta-hierarchy Dynamics Unit, NIFS

① 曲率を持つ磁場における一般化長谷川三間方程式の開発

A Generalized Hasegawa-Mima Equation for Drift Wave Turbulence in Curved Magnetic Fields

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Introduction

The Hasegawa-Mima (HM) equation [3] describes 2-dimensional electrostatic plasma turbulence in a straight homogeneous magnetic field $\boldsymbol{B} = B_0 \nabla z$. Defining $[f,g] = f_x g_y - f_y g_x$, $\nabla_{(x,y)} = \nabla x\,\partial_x + \nabla y\,\partial_y$ and $\Delta_{(x,y)} = \partial_x^2 + \partial_y^2$, the equation is

$$\frac{\partial}{\partial t}\left(\lambda\varphi - \frac{\sigma}{B_0^2}\Delta_{(x,y)}\varphi\right) = \frac{\sigma}{B_0^3}\left[\varphi, \Delta_{(x,y)}\varphi\right] + \frac{\beta}{B_0}\varphi_y, \qquad \lambda = \frac{e}{k_B T_e}, \qquad \sigma = \frac{m}{Ze}. \tag{1}$$

In a domain $\Sigma \subset \mathbb{R}^2$, the invariants of the HM equation are mass M, energy H, and generalized enstrophy W:

$$M_\Sigma = \int_\Sigma \varphi\, dxdy, \qquad H_\Sigma = \frac{1}{2}\int_\Sigma \left(\lambda\varphi^2 + \sigma\left|\nabla_{(x,y)}\varphi\right|^2\right) dxdy,$$
$$W_\Sigma = \frac{1}{2}\int_\Sigma \left[\lambda\left|\nabla_{(x,y)}\varphi\right|^2 + \frac{\sigma}{B_0^2}\left(\Delta_{(x,y)}\varphi\right)^2\right] dxdy. \tag{2}$$

- A closed equation for the electric potential describing electrostatic plasma turbulence in a general (inhomogeneous and curved) magnetic field is not available at present.

- The purpose of this study is to generalize (1) to an arbitrary magnetic field $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x}) \neq \boldsymbol{0}$.

[3] A. Hasegawa & K. Mima, Phys. Fluids **21**, 1, pp. 87-92 (1977).

Two-fluid Hasegawa-Mima Ordering

Order	Dimensionless	Fields	Distances	Rates	Velocities
1		$\boldsymbol{B}, A_e$	L	ω_c	
ϵ	$\lambda\varphi, \omega_c^{-1}\partial_t, L\nabla \log B, L\nabla \log A_e$	$\boldsymbol{E}_\perp$		$\tau_d^{-1}, \boldsymbol{v}_{\boldsymbol{E}}/L$	$\boldsymbol{v}_{\boldsymbol{E}}$
ϵ^2	τ_d/τ_b			$\boldsymbol{v}_{\mathrm{pol}}/L$	$\boldsymbol{v}_{\mathrm{pol}}$
ϵ^3		$E_\parallel, P$		$\tau_b^{-1}, v_\parallel/L$	$v_\parallel$

- Slow parallel dynamics, thermalized electrons, and cold ions
- Small magnetic field and electron density gradients $L\nabla \log B \sim L\nabla \log A_e \ll 1$ over the maximum turbulence scale $L \sim k_{\perp\mathrm{min}}^{-1}$

Experimental observations in dipole magnetic fields [4,5]

- Existence of drift wave turbulence and zonal flows in systems where both electron spatial density and magnetic field have strong gradients over spatial scales comparable to that of electric field and density fluctuations (entropy modes [6])

[4] A. C. Boxer et al. Nat. Phys. **6**, pp. 207-212 (2010) [5] N. Kenmochi et al. Nuc. Fusion **62** 026041 (2022) [6] D. T. Garnier et al. PoP **24** 012506 (2017).

Literature

(i) **Sugama & Horton PoP 1998**
- Gyrokinetic ordering
- Slab + Cylindrical
- Strong background flow $\boldsymbol{V}_0$

(ii) **Brizard Phys. Fluids 1992**
- Gyrokinetic ordering
- Small magnetic field and density gradients

(iii) **Frieman & Chen Phys. Fluids 1982**
- Gyrokinetic ordering
- Symmetric magnetic field
- Fourier form

Our aim: derive generalized HM eq. with

A. Two-fluid ordering (no need for ρ)
B. No conditions on geometry of $\boldsymbol{B}$ or A_e

$$\begin{aligned}&\left[\frac{\partial}{\partial t_0}+\left\{\mathbf{V}_0+\frac{c}{Z_i e B}\mathbf{b}\times(Z_i e B\nabla\Phi_1+m_i\mathbf{V}_0\cdot\nabla\mathbf{V}_0)\right\}\cdot\nabla\right]\\&\quad\times\left[(1-\rho_s^2\nabla_\perp^2)\left(\frac{e\hat{\phi}(\mathbf{x})}{T_e}\right)\right]+\frac{cT_e}{eB}\left[-\mathbf{b}\times\nabla\ln n_e+B\nabla\right.\\&\quad\left.\times\left(\frac{\mathbf{b}}{B}\right)\right]\cdot\nabla_\perp\left(\frac{e\hat{\phi}(\mathbf{x})}{T_e}\right)\\&=c_s\rho_s^3\mathbf{b}\times\nabla\left(\frac{e\hat{\phi}(\mathbf{x})}{T_e}\right)\cdot\nabla\nabla_\perp^2\left(\frac{e\hat{\phi}(\mathbf{x})}{T_e}\right)\end{aligned}\qquad(37)$$

$$\begin{aligned}(\tau^{-1}-\bar{\nabla}_\perp^2)\bar{\partial}_t\varphi&=[\bar{\omega}_*-2\eta_B\bar{\omega}_\kappa+\bar{\omega}_*(1+\eta_i)\bar{\nabla}_\perp^2]\varphi\\&\quad-\bar{\nabla}_\perp\cdot\{\bar{\nabla}_\perp\varphi,(\varphi+p_{\perp i})\}\\&\quad-\bar{\partial}_\parallel^* v_{\parallel i}-\eta_B\bar{\omega}_\kappa(p_{\perp i}+p_{\parallel i}),\end{aligned}\qquad(62)$$

$$\begin{aligned}&\left((1+k_\perp^2\rho_s^2)\frac{\partial}{\partial t}-i\mathbf{k}_\perp\cdot\bar{\mathbf{v}}_{de}+i\omega_{*e}\right)\delta\psi_n\\&\quad=\pi C_s^2\sum_{\substack{n'+n''\\=n}}W_{n',n''}\sum_p\exp(-in''2\pi pQ)\tilde{C}_{n',n''}\\&\quad\times\int_{-\infty}^{\infty}d\hat{\theta}_{n'}\,\delta(\hat{\theta}_n-\hat{\theta}_{n'})\int_{-\infty}^{\infty}d\hat{\theta}_{n''}\,\delta(\hat{\theta}_{n''}-\hat{\theta}_n-2\pi p)\\&\quad\times\rho_s^2[(k_\perp'')^2-(k_\perp')^2]\delta\psi_{n'}\,\delta\psi_{n''},\end{aligned}\qquad(75)$$

Generalized Hasegawa-Mima (GHM) Equation

We propose [1,7] a generalization of the HM equation which accounts for drift wave turbulence in systems with strong magnetic field and electron density gradients $L\nabla \log B \sim L\nabla \log A_e \sim 1$:

$$\frac{\partial}{\partial t}\left[\lambda A_e \chi - \sigma \nabla \cdot \left(\frac{A_e \nabla_\perp \chi}{B^2}\right)\right] = \nabla \cdot \left[A_e \left(\sigma \frac{\boldsymbol{B} \cdot \nabla \times \boldsymbol{v}_{\boldsymbol{E}}^{\chi}}{B^2} - 1\right) \boldsymbol{v}_{\boldsymbol{E}}^{\chi}\right]. \tag{3}$$

- $\boldsymbol{x} \in \Omega \subseteq \mathbb{R}^3, t \in [0, \infty)$
- $\chi(\boldsymbol{x}, t) = \varphi(\boldsymbol{x}, t) + \frac{\sigma}{2} \boldsymbol{v}_{\boldsymbol{E}}^2(\boldsymbol{x}, t)$ (charged particle energy)
- $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x}) \neq \boldsymbol{0}$ (static magnetic field of arbitrary geometry)
- $A_e = A_e(\boldsymbol{x})$ (leading order electron spatial density, $n_e = A_e \exp(e\varphi/k_B T_e)$)
- $\nabla_\perp = -B^{-2} \boldsymbol{B} \times (\boldsymbol{B} \times \nabla)$
- $\boldsymbol{v}_{\boldsymbol{E}}^{\chi} = \boldsymbol{B} \times \nabla \chi / B^2$
- $\boldsymbol{v}_{\boldsymbol{E}} = \boldsymbol{B} \times \nabla \varphi / B^2$

HM eq. when
$\boldsymbol{B} = B_0 \nabla z$,
$\log A_e = \log A_{e0} + \beta x$,
$\beta L \sim \epsilon$

The ordering leading to (3) does not involve conditions on spatial derivatives of $\boldsymbol{B}$ or A_e:

➢ The GHM equation (3) can sustain turbulence over spatial scales comparable to those of magnetic field and electron spatial density, as well as over distances of the order of the ion gyroradius and smaller.

[1] N. Sato and M. Yamada J. Plasma Phys. **88** 3 (2022) [7] N. Sato and M. Yamada arXiv:2306.16660 (2023).

Two Fluid Generalized Hasegawa-Mima Ordering

Order	Dimensionless	Fields	Distances	Rates	Velocities
1		$\boldsymbol{B}, A_e$	L	ω_c	
ϵ	$\lambda\varphi, \omega_c^{-1}\partial_t$	$\boldsymbol{E}_\perp$		$\tau_d^{-1}, \boldsymbol{v}_{\boldsymbol{E}}/L$	$\boldsymbol{v}_{\boldsymbol{E}}$
ϵ^2	τ_d/τ_b			$\boldsymbol{v}_{\rm pol}/L$	$\boldsymbol{v}_{\rm pol}$
ϵ^3		$E_\parallel, P$		$\tau_b^{-1}, v_\parallel/L$	$v_\parallel$

- The GHM equation (3) can be obtained from a two fluid plasma model with cold ions and hot thermalized electrons with the aid of the ordering above [1,7].
- No ordering conditions apply to spatial derivatives of $\boldsymbol{B}$ or A_e.
- In particular, the GHM equation (3) is compatible with the following configurations

$$k_\perp \sim \rho_i^{-1}, \qquad k_\perp \sim \nabla \log B \sim \nabla \log A_e$$

No limit to the magnitude of $k_\perp$

$k_\perp$, $\nabla \log B$, and $\nabla \log A_e$ can be comparable (not possible with standard HM equation)

[1] N. Sato and M. Yamada J. Plasma Phys. **88** 3 (2022) [7] N. Sato and M. Yamada arXiv:2306.16660 (2023).

Guiding Center Generalized Hasegawa-Mima Ordering

Order	Dimensionless	Fields	Distances	Rates	Velocities
ϵ^{-1}		$\boldsymbol{B}, \boldsymbol{E}_\perp$		ω_c	
1		$\boldsymbol{E}_\parallel$	L	$\boldsymbol{v}/L, \boldsymbol{v}_{\boldsymbol{E}}/L, \tau^{-1}$	$\boldsymbol{v}, \boldsymbol{v}_{\boldsymbol{E}}$
ϵ	$\rho_i/L, (\omega_c\tau_d)^{-1}$		ρ_i	$\boldsymbol{v}_\nabla/L, \boldsymbol{v}_\kappa/L, \boldsymbol{v}_{\mathrm{pol}}/L$	$\boldsymbol{v}_\nabla, \boldsymbol{v}_\kappa, \boldsymbol{v}_{\mathrm{pol}}$

Guiding center ordering required for the existence of the first adiabatic invariant μ.

Order	Dimensionless	Fields	Distances	Rates	Velocities
ϵ^{-1}		$\boldsymbol{B}, \boldsymbol{E}_\perp$		ω_c	
1		A_e	L	$\tau_d^{-1}, \boldsymbol{v}_{\boldsymbol{E}}/L$	$\boldsymbol{v}_{\boldsymbol{E}}$
ϵ	$\lambda\varphi, \rho_i/L, (\omega_c\tau_d)^{-1}, k_B T_c/\frac{m}{2}\boldsymbol{v}_{\boldsymbol{E}}^2$		ρ_i	$\boldsymbol{v}_{\mathrm{pol}}/L$	$\boldsymbol{v}_{\mathrm{pol}}$
ϵ^2	$\frac{m}{2}\boldsymbol{v}_{\boldsymbol{E}}^2/k_B T_e, \tau_d/\tau_b$	$E_\parallel{}'$		$\boldsymbol{v}_\nabla/L, u/L, \tau_b^{-1}$	u
ϵ^5				$\boldsymbol{v}_\kappa/L$	$\boldsymbol{v}_\kappa$

Drift wave turbulence ordering for the derivation of the GHM eq. (3) in guiding-center theory [6].

[7] N. Sato and M. Yamada arXiv:2306.16660 (2023).

Invariants of the GHM Equation

Invariant	Expression	Field Conditions	Boundary Conditions
Mass M_Ω	$\int_\Omega A_e(1+\lambda\chi)d\boldsymbol{x}$	None	$A_e\boldsymbol{V}_{dw}\cdot\boldsymbol{n}=0$
Energy H_Ω	$\frac{1}{2}\int_\Omega A_e\left(\lambda\chi^2+\sigma\frac{\lvert\nabla_\perp\chi\rvert^2}{B^2}\right)d\boldsymbol{x}$	None	$A_e\chi\boldsymbol{V}_{dw}\cdot\boldsymbol{n}=0$
Enstrophy W_Ω	$\int_\Omega A_e w\, d\boldsymbol{x}$	$\nabla\times\left(A_e\frac{\boldsymbol{B}}{B^2}\right)=\boldsymbol{0}$	$wA_e\boldsymbol{v}_{\boldsymbol{E}}^{\chi}\cdot\boldsymbol{n}=0$

- $\chi(\boldsymbol{x},t)=\varphi(\boldsymbol{x},t)+\frac{\sigma}{2}\boldsymbol{v}_{\boldsymbol{E}}^2(\boldsymbol{x},t)$ (charged particle energy)
- $w=w(\lambda\chi-\sigma\omega/A_e)$ (arbitrary function of argument)
- $\omega=\nabla\cdot(A_eB^{-2}\nabla_\perp\chi)$ (vorticity)
- $\boldsymbol{V}_{dw}=\left(1-\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_{\boldsymbol{E}}^{\chi}}{B^2}\right)\boldsymbol{v}_{\boldsymbol{E}}^{\chi}-\sigma\frac{\nabla_\perp\chi_t}{B^2}$ (effective drift velocity)
- $\boldsymbol{v}_{\boldsymbol{E}}^{\chi}=\boldsymbol{B}\times\nabla\chi/B^2$

[7] N. Sato and M. Yamada arXiv:2306.16660 (2023).

Zonal Flows and Drift Waves in Dipole Magnetic Fields

- In an axially symmetric dipole magnetic field $\boldsymbol{B} = \nabla\zeta(r,z) = \nabla\Psi(r,z) \times \nabla\phi$ the GHM eq. (3) reduces to

$$\frac{\partial}{\partial t}\left[\lambda A_e\chi - \sigma\nabla\cdot\left(A_e\frac{\nabla_\perp\chi}{B^2}\right)\right] = B^2\left[\chi, \frac{A_e}{B^2}\left(\sigma\frac{\Delta_\perp\chi}{B^2} - 1\right)\right]_{(\Psi,\phi)}, \qquad [f,g]_{(\Psi,\phi)} = f_\Psi g_\phi - f_\phi g_\Psi. \quad (4)$$

Steady solutions χ_0 of (4) satisfy

$$\frac{A_e}{B^2}\left(\sigma\frac{\Delta_\perp\chi_0}{B^2} - 1\right) = f(\chi_0,\zeta). \quad (5)$$

Under suitable boundary conditions, eq. (5) admits zonal flow solutions $\chi_{0\phi} = 0$ such that $\boldsymbol{v}_{\boldsymbol{E}}^{\chi} = \chi_{0\Psi}(r,z)\partial_\phi$.

- Given $\omega \in \mathbb{R}, \ell \in \mathbb{Z}$ drift waves $\chi_d = \xi(\zeta,\Psi)\exp\{-\mathrm{i}(\ell\phi + \omega t)\}$ exist with ξ solution of

$$\omega = \frac{\ell\frac{\partial}{\partial\Psi}\log\left(\frac{A_e}{B^2}\right)}{\frac{\sigma\ell^2}{r^2B^2} + \lambda - \frac{\sigma}{A_e\xi}\nabla\cdot\left(A_e\frac{\nabla_\perp\xi}{B^2}\right)} \xRightarrow[\text{HM Limit}]{} \omega = -\frac{k_y\beta}{\lambda B_0 + \sigma\frac{k_x^2 + k_y^2}{B_0}}. \quad (6)$$

[8] N. Sato and M. Yamada arXiv:2305.16668 (2023).

Hamiltonian Structure of the GHM Equation

The GHM eq. (3) is endowed with an antisymmetric bracket structure:

$$\frac{\partial \eta}{\partial t} = \{\eta, H_\Omega\}, \qquad \{F,G\} = \int_\Omega A_e \left(1 - \sigma \frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_E^\chi}{B^2}\right) \nabla F_\eta \cdot \frac{\boldsymbol{B}}{B^2} \times \nabla G_\eta \, d\boldsymbol{x}\,, \qquad \eta = \lambda A_\mathrm{e}\chi - \sigma\omega. \tag{7}$$

The GHM eq. (3) is endowed with a Poisson bracket provided that the magnetic field satisfies the integrability condition $\nabla \times (A_e \boldsymbol{B}/B^2) = \boldsymbol{0}$. The Poisson bracket is:

$$\{F,G\}_{\mathrm{PB}} = \int_\Omega A_e \nabla F_\eta \cdot \frac{\boldsymbol{B}}{B^2} \times \nabla G_\eta \, d\boldsymbol{x}\,. \tag{8}$$

- The integrability condition also guarantees the conservation of enstrophy W_Ω, which is a Casimir of $\{F,G\}_{\mathrm{PB}}$.
- Analogy with ExB dynamics: $\dot{\boldsymbol{x}} = \boldsymbol{E}\times\boldsymbol{B}/B^2$ Hamiltonian iff $\boldsymbol{B}\cdot\nabla\times\boldsymbol{B} = 0$.
- The Poisson bracket (8) reduces to the HM and 2D ideal fluid Poisson brackets.

[8] N. Sato and M. Yamada arXiv:2305.16668 (2023).

Nonlinear Stability of Steady Solutions of the GHM Equation

Theorem 1. (Nonlinear stability of steady solutions of the GHM equation) *Let $\chi_0(\boldsymbol{x}) \in C^2(\Omega)$ denote a critical point of the energy-Casimir function $\mathfrak{H}_\Omega = H_\Omega + \gamma M_\Omega + \nu W_\Omega$. If the condition $\nabla \times (A_e\, \boldsymbol{B}/B^2) = \boldsymbol{0}$ holds, assume that the function $w(\eta/A_e)$ appearing within the integrand of the Casimir invariant W_Ω is twice differentiable in its argument, and that it satisfies*

$$0 < c_m \leq \nu w'' = \nu \frac{d^2 w}{d(\eta/A_e)^2} \leq c_M < \infty,$$

with c_m and c_M real constants. If $\nabla \times (A_e\, \boldsymbol{B}/B^2) \neq \boldsymbol{0}$ set $\nu = 0$. Further assume that $\boldsymbol{B}, A_e \in C^2(\bar{\Omega})$, that their minima satisfy $B_m, A_{em} > 0$, and that the GHM eq. (3) admits a solution $\chi(\boldsymbol{x}, t) \in C^2\big(\Omega \times [0, t)\big)$ for all $t \geq 0$ such that $\delta\chi = \chi - \chi_0$ and $A_e = 0$ on the boundary $\partial\Omega$. Then, the critical point χ_0 is nonlinearly stable: there exists a positive real constant $\mathcal{C}$ such that

$$\|\chi(t) - \chi_0\|_\perp^2 \leq \mathcal{C}\|\chi(0) - \chi_0\|_\perp^2 \quad \forall t \geq 0,$$

with

$$\|\chi\|_\perp^2 = \begin{cases} \|\chi\|_{L^2(\Omega)}^2 + \|\nabla\chi\|_{L^2(\Omega)}^2 + \|\eta\|_{L^2(\Omega)}^2 & \text{if} \quad \nabla \times (A_e\, \boldsymbol{B}/B^2) = \boldsymbol{0}, \\ \|\chi\|_{L^2(\Omega)}^2 + \|\nabla\chi\|_{L^2(\Omega)}^2 & \text{if} \quad \nabla \times (A_e\, \boldsymbol{B}/B^2) \neq \boldsymbol{0}, \end{cases}$$

where $L^2(\Omega)$ denotes the standard L^2 norm in Ω and we used the abbreviated notation $\chi(t) = \chi(\boldsymbol{x}, t)$.

Remark 1. Theorem 1 generalizes Arnold's result [9] concerning the stability of a 2D ideal fluid, which corresponds to $\boldsymbol{B} = \nabla z$, $A_e = \sigma = 1$, and $\lambda = 0$.

[8] N. Sato and M. Yamada arXiv:2305.16668 (2023). [9] V. I. Arnold and B. A. Khesin, in Topological methods in hydrodynamics, Springer, pp. 89-96 (1998).

Concluding Remarks – Summary

- We have derived a model GHM equation (3) describing electrostatic plasma turbulence in a general magnetic field from both two-fluid and guiding center theories.
- The ordering leading to (3) does not involve conditions on spatial derivatives of $\boldsymbol{B}$ or A_e: the GHM equation can sustain turbulence over spatial scales comparable to those of magnetic field and electron spatial density, as well as over distances of the order of the ion gyroradius.
- The GHM equation reduces to the HM equation in the limit of a straight magnetic field.
- The GHM equation exhibits both zonal flows and drift waves in dipole geometry.
- The equation preserves mass M_Ω and energy H_Ω. Enstrophy W_Ω appears as a Casimir invariant of a Poisson bracket when $\nabla \times (A_e \boldsymbol{B}/B^2) = \boldsymbol{0}$.
- We have obtained a nonlinear stability criterion for steady solutions that generalzes Arnold's classical result.

② 三次元閉じ込め磁場の存在問題に関する数理解析

Existence of Three-Dimensional Magnetohydrostatic Equilibria

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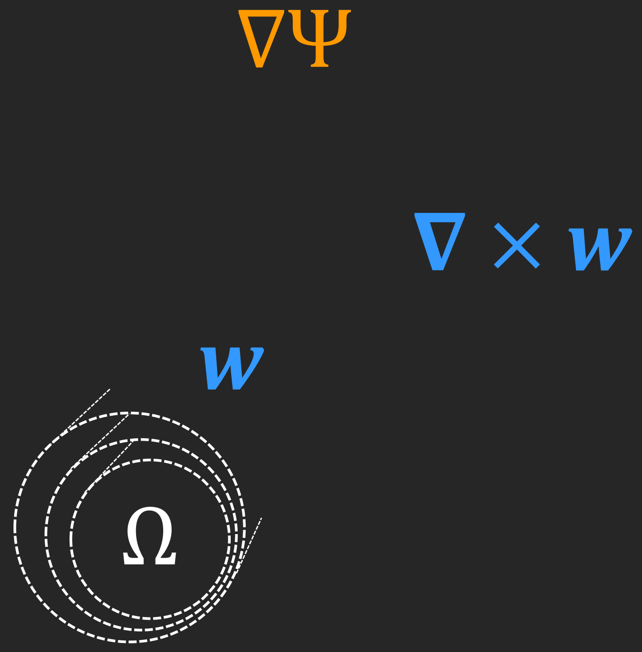
Introduction

This study is concerned with the equation

$$[(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}] \times \nabla\Psi = \boldsymbol{0}, \qquad \nabla \cdot \boldsymbol{w} = 0 \qquad \text{in } \Omega. \tag{1}$$

Here, the unknown $\boldsymbol{w}\,(\boldsymbol{x})$ is a three-dimensional vector field with Cartesian components w_i, $i = 1,2,3$, defined in a smooth toroidal domain $\Omega \subset \mathbb{R}^3$ foliated by nested toroidal surfaces corresponding to level sets of a function $\Psi(\boldsymbol{x})$ such that the bounding surface is given by $\partial\Omega = \{\boldsymbol{x} \in \mathbb{R}^3 \colon \Psi = \Psi_0 \in \mathbb{R}\}$.



- Given a set of nested toroidal surfaces Ψ in Ω, can one always find a solenoidal vector field $\boldsymbol{w}$ such that both $\boldsymbol{w}$ and $\nabla \times \boldsymbol{w}$ are tangent to the level sets of Ψ?

- More generally, both Ψ and Ω can be considered as variables of the problem.

Relation with Fluid Mechanics and Magnetohydrodynamics (MHD)

In the context of fluid mechanics, (1) represents a generalization of a more difficult equation

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = \nabla \Psi, \qquad \nabla \cdot \boldsymbol{w} = 0 \qquad \text{in } \Omega, \tag{2}$$

where $\boldsymbol{w} = \boldsymbol{u}$ is the fluid velocity of an incompressible steady Euler flow, $\nabla \times \boldsymbol{w} = \boldsymbol{\omega}$ the vorticity, and $P = -\Psi - \frac{1}{2}\boldsymbol{u}^2$ the fluid pressure [1].

Equation (2) also corresponds to the MHD equilibrium equation with $\boldsymbol{w} = \boldsymbol{B}$ the magnetic field, $\nabla \times \boldsymbol{w} = \boldsymbol{J}$ the electric current, and $P = \Psi$ the pressure field. Its solution in a toroidal domain is crucial for the design of confining magnetic fields in nuclear fusion reactors [2].

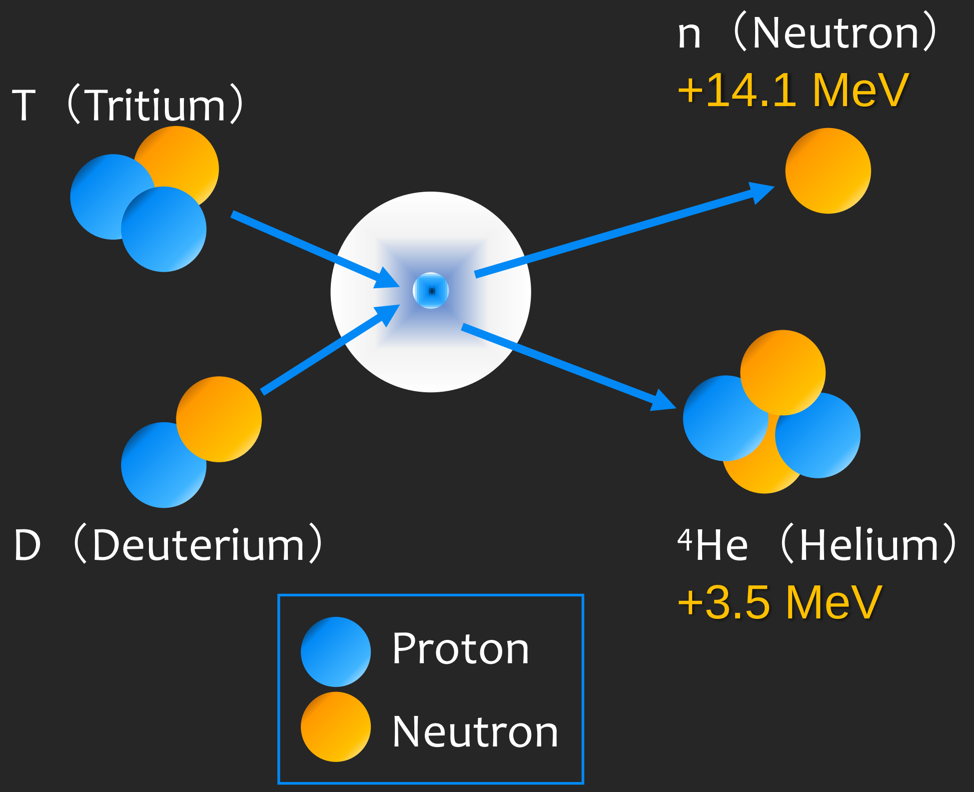
- Any solution of (2) is also a solution of (1).
- Physically, (1) can be related to flows with anisotropic pressure or anisotropic MHD equilibria, i.e., $\nabla \Psi \rightarrow \nabla \cdot \Pi$.

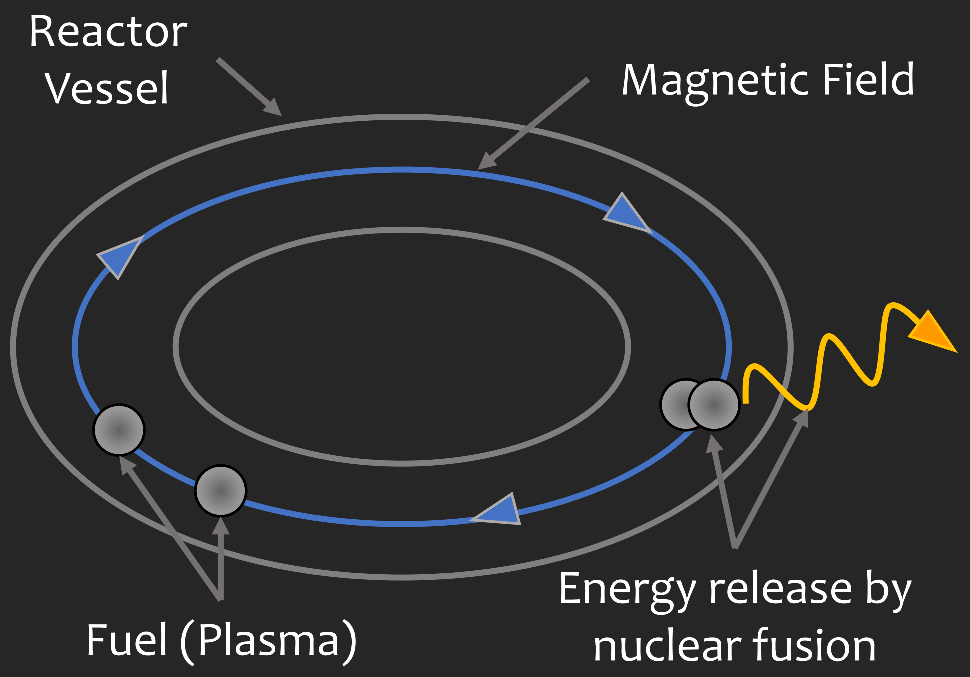
[1] H. K. Moffatt, J. Fluid Mech. **159**, pp. 359-378 (1985). [2] M. D. Kruskal and R. M. Kulsrud, The Physics of Fluids 1, 4 (1958).

Magnetic Confinement Fusion

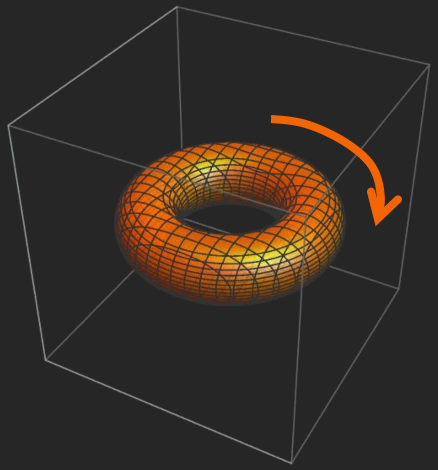
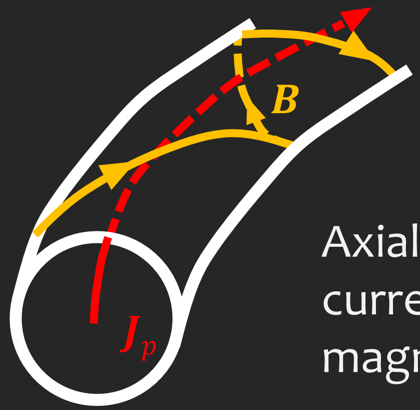
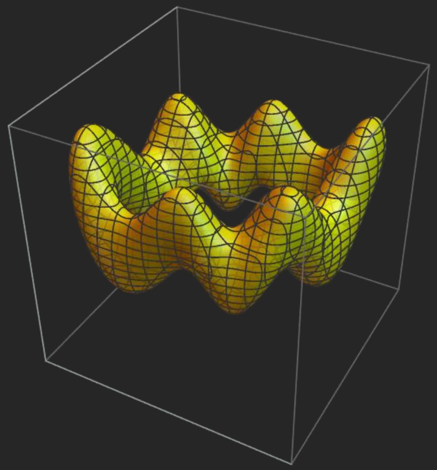
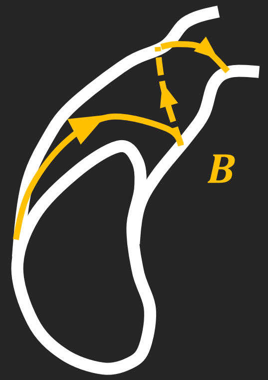
Nuclear fusion is the nuclear reaction powering stars. It represents an attractive carbon-free source of energy: hydrogen contained in 1 litter of water is enough to provide energy to a standard household for 1 year.

In a fusion reaction, part of the mass of the fusing nuclei is converted into kinetic energy of reaction products. In a fusion reactor, due to high temperatures ($\sim 10^8$ K) the fuel (plasma) is confined via a magnetic field within a toroidal vessel.





Tokamaks & Stellarators

	Reactor Shape	Field Line Twist Mechanism		Properties
Axial Symmetry (Tokamak)			Axial plasma current twists magnetic field lines	+ Conservation of angular momentum improves radial confinement − Plasma current is prone to instability
Asymmetry (Stellarator)			Suitably crafted external coils generate a twisted magnetic field	+ Reduced plasma current implies better steady operation − Confinement is degraded by loss of conserved angular momentum

In both cases, the magnetic field is governed by (2). An additional condition (quasisymmetry) is needed for stellarators to restore a conserved momentum

Background on Equation (2): An Unsolved PDE Problem

A general theory concerning the existence of solutions of (2) is not available [3]: it is not known whether regular steady fluid flows or equilibrium magnetic fields exist in a bounded domain Ω of arbitrary shape.

The intrinsic mathematical difficulty behind equation (2) can be understood in terms of characteristic surfaces. If considered as a system of nonlinear first order PDEs for the unknowns $\boldsymbol{w}$, Ψ, the characteristic surfaces S of equation (2) are determined by the characteristic equation [4]

$$(\nabla S)^2(\boldsymbol{w}\cdot\nabla S)^2 = 0. \tag{3}$$

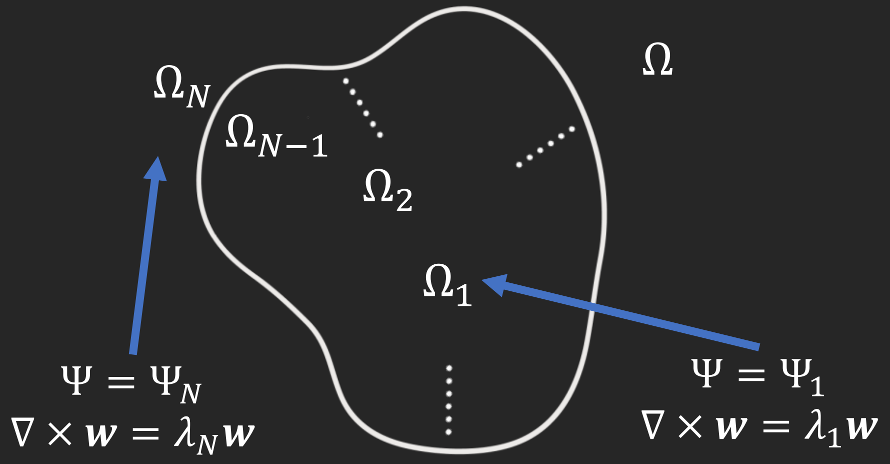
- Equation (2) has a mixed behavior (twice elliptic and twice hyperbolic), with the nontrivial characteristic surfaces $(\boldsymbol{w}\cdot\nabla S)^2 = 0$ associated with hyperbolicity depending on the unknown $\boldsymbol{w}$.
- These features make (2) one of the hardest PDEs in mathematical physics.
- Despite its difficulty, steady progress has been made in the understanding of equation (2) since the inception of the problem in the early years of magnetic confinement fusion research.

[3] C. Lo Surdo, Int. J. Math. Math. Sci. **9**, pp. 123-130 (1986). [4] Z. Yoshida and H. Yamada, Prog. Theor. Phys. **84** 2 (1990).

Background on Equation (2): Weak Solutions

Given a nondegenerate toroidal domain $\Omega_1 \subset \mathbb{R}^3$, one can construct a piecewise smooth solution $(\boldsymbol{w}, \Psi)$ of (2) with constant pressure $\Psi = \Psi_i$ in each subdomain Ω_i, $i = 1, \ldots, N$, with $\overline{\Omega}_{i-1} \subset \Omega_i \subset \overline{\Omega}$, $2 \leq i \leq N$, nested toroidal domains [5]. See also [6].

In each Ω_i eq. (2) reduces to the eigenvalue problem for the curl operator [7]. The eigenvectors $\boldsymbol{w}$ such that $\nabla \times \boldsymbol{w} = \lambda_i \boldsymbol{w}$ are called Beltrami fields.



The solutions are weak, in the sense that $(\boldsymbol{w}, \Psi) \in L^2(\Omega)$ with

$$\int_\Omega \left[\boldsymbol{w} \cdot (\boldsymbol{w} \cdot \nabla)\boldsymbol{v} - \left(\Psi + \frac{1}{2}\boldsymbol{w}^2\right)\nabla \cdot \boldsymbol{w}\right] dV = 0, \qquad \int_\Omega \boldsymbol{w} \cdot \nabla\varphi \, dV = 0, \qquad \forall \boldsymbol{v} \in C_c^1(\Omega), \varphi \in C^1(\Omega). \quad (4)$$

In a slightly different setting where Ψ is not required to be constant on $\partial\Omega$, nontrivial strong solutions of (2) in the class $H^1(\Omega)$ have been reported in [8]. These solutions are obtained as steady states of a Voigt approximation scheme of the time-dependent viscous non-resistive incompressible magnetohydrodynamics equations in the limit $t \to \infty$.

[5] A. Enciso, A. Luque, and D. Peralta-Salas, arXiv:2104.08149v3 (2022). [6] O. P. Bruno and P. Laurence, Commun. Pure Appl. Math. **49**, pp. 717-764 (1996). [7] Z. Yoshida and Y. Giga, Math. Z. **204**, pp. 235-245 (1990). [8] P. Constantin and F. Pasqualotto, arXiv:2208.11109v1 (2022).

Background on Equation (2): Continuous Euclidean Isometries and the Grad-Shafranov Equation

Eq. (2) is greatly simplified when $\boldsymbol{w}$ and Ψ are invariant under a continuous Euclidean isometry, i.e., a continuous transformation of $\mathbb{R}^3$ that preserves the Euclidean distance $d\boldsymbol{x}^2 = dx^2 + dy^2 + dz^2$ between points:

$$\mathcal{L}_{\boldsymbol{\eta}}\, d\boldsymbol{x}^2 = 0 \quad \leftrightarrow \quad \boldsymbol{\eta} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}, \qquad \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3. \tag{5}$$

In plasma physics, invariance under a continuous Euclidean isometry is referred to as a symmetry of the system. In formulae, $\boldsymbol{w}$ and Ψ are symmetric whenever constant vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ with $\boldsymbol{a}^2 + \boldsymbol{b}^2 \neq 0$ exist such that

$$\mathcal{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}} \boldsymbol{w} = \boldsymbol{0}, \qquad \mathcal{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}} \Psi = 0. \tag{6}$$

When (6) holds, eq. (2) reduces to the Grad-Shafranov equation [9, 10], a nonlinear 2nd order elliptic PDE for the unknown Θ, with $\Psi = \Psi(\Theta)$. Regular (symmetric) solutions of the Grad-Shafranov equation can be obtained by elliptic theory. Taking coordinates $\boldsymbol{y}$ and setting $\boldsymbol{w} = \nabla\Theta \times \nabla y^3 + w^3 \partial_3$ with $\partial_3 = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$, the equation is:

$$\Delta\Theta - \nabla\Theta \cdot \nabla \log g_{33} - g_{33}\frac{d\Psi}{d\Theta} + w^3 \frac{dw_3}{d\Theta} = 0 \quad \text{in } \Omega, \qquad \Theta = \Theta_0 \quad \text{on } \partial\Omega. \tag{7}$$

[9] J. W. Edenstrasser, J. Plasma Phys. **24**, pp. 299-313 (1980). [10] J. W. Edenstrasser, J. Plasma Phys. **24**, pp. 515-518 (1980).

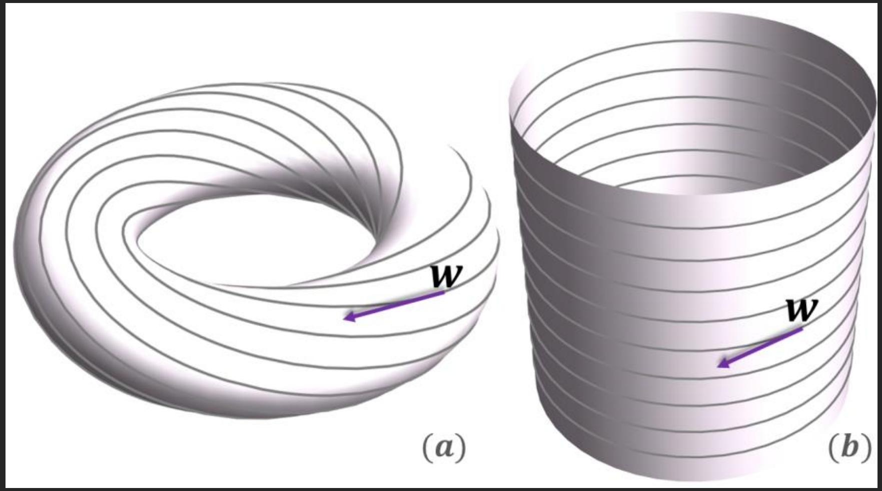
Background on Equation (2): Arnold's Structure Theorem

Stellarators sacrifice axial symmetry in favor of field line twist aimed at minimizing plasma losses at the vessel boundary $\partial\Omega$ caused by cross-field dynamics of charged particles [11].

Note that even if asymmetric solutions of (2) exist, they will not necessarily work as confining magnetic fields, because other requirements, such as quasisymmetry [12] and a small electric current, must be enforced on $\boldsymbol{w}$.

According to the Grad conjecture [13], only "configurations of great geometrical symmetry" result in well behaved equilibria. This is understood as eq. (4) being necessary for the existence of regular solutions of (2).

Arnold's structure theorem [14] characterizes the topology of any analytic solution of (2) such that $\boldsymbol{w}$ and $\nabla \times \boldsymbol{w}$ are not everywhere collinear: when (2) is considered in a connected analytic bounded domain Ω together with tangential boundary conditions $\boldsymbol{w} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$, any contour of Ψ that does not intersect the boundary $\partial\Omega$ and such that $\nabla\Psi \neq \boldsymbol{0}$ is a two-dimensional torus.



Note: level sets of Ψ cannot be spherical due to the hairy ball theorem [15], which precludes the existence of a continuous non-vanishing vector field always tangent to a 2-sphere.

[11] P. Helander, Rep. Prog. Phys. **77**, 087001 (2014). [12] E. Rodriguez, P. Helander, and A. Bhattacharjee, Phys. Plasmas **27**, 062501 (2020). [13] H. Grad, The Physics of Fluids **10**, 1 (1967). [14] V. I. Arnold, J. Appl. Math. Mech. **30**, pp. 223-226 (1966). [15] M. Eisenmberg and R. Guy, Am. Math. Mon. **86**, pp. 571-574 (1979).

Equation (1) and Anisotropic Pressure

In (2), the scalar Ψ can be generalized with a tensor Π. One obtains anisotropic MHD equilibria:

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = \nabla \cdot \Pi, \qquad \nabla \cdot \boldsymbol{w} = 0 \qquad \text{in } \Omega, \qquad \Pi^{ij} = \left(P - \frac{1}{2}\gamma \boldsymbol{w}^2\right)\delta^{ij} + \gamma w^i w^j. \tag{8}$$

Equation (8) can be written as:

$$(1-\gamma)(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = \nabla P - \frac{1}{2}\boldsymbol{w}^2 \nabla\gamma + (\boldsymbol{w} \cdot \nabla\gamma)\boldsymbol{w}, \qquad \nabla \cdot \boldsymbol{w} = \boldsymbol{0} \qquad \text{in } \Omega. \tag{9}$$

Assume $\boldsymbol{w} = \nabla\Psi \times \nabla\Theta$, $\boldsymbol{w} \cdot \nabla \sim 0$, $P_\Theta = \boldsymbol{w}^2\gamma_\Theta/2$, and $\lambda = P_\Psi - \boldsymbol{w}^2\gamma_\Psi/2$. Equation (9) becomes:

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = \lambda \nabla\Psi, \qquad \nabla \cdot \boldsymbol{w} = \boldsymbol{0} \qquad \text{in } \Omega, \tag{10}$$

which corresponds to equation (1).

Equation (1) as a PDE Problem

- Considering the challenge posed by equation (2), here we examine the simplified problem of equation (1).
- While in (2) the magnitude of $(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}$ along $\nabla\Psi$ is exactly $|\nabla\Psi|$, no such requirement appears in (1).
- Eq. (2) simplifies the mathematical difficulty by a 'half', since the governing equations are reduced from 2 to 1.
- Any conditions preventing the existence of solutions of (1) would also apply to (2).
- If regular solutions of (1) could be obtained, it would be possible to identify the geometrical obstruction preventing such solutions from solving (2) as well.
- **Strategy**: reduce (1) by a Clebsch representation [16, 17] of $\boldsymbol{w}$ by a pair of Clebsch potentials (Ψ, Θ) that reflect the foliated $(\nabla\Psi \cdot \boldsymbol{w} = 0)$ and solenoidal $(\nabla \cdot \boldsymbol{w} = 0)$ nature of the candidate solution $\boldsymbol{w} = \nabla\Psi \times \nabla\Theta$.
- **Merit**: the topology of the foliation associated with the (given) function Ψ can be enforced a priori, leaving the analysis of the existence of solutions as an independent issue for Θ.

[16] Z. Yoshida, J. Math. Phys. **50**, 113101 (2009). [17] Z. Yoshida and P. J. Morrison, Phys. Rev. Lett. **119**, 244501 (2017).

Main Result

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ denote a bounded domain. Assume that the bounding surface $\partial\Omega$ is a hollow torus corresponding to two distinct level sets of a smooth function $\Psi \in C^\infty(\Omega)$, with $\nabla\Psi \neq \mathbf{0}$ in Ω, and that the level sets of Ψ foliate Ω with nested toroidal surfaces endowed with angle coordinates μ, ν with smooth gradients $\nabla\mu, \nabla\nu \in C^\infty(\Omega)$. Then, the system of partial differential equations*

$$[(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}] \times \nabla\Psi = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{w} = 0 \qquad \text{in } \Omega,$$

admits a nontrivial solution $\boldsymbol{w} \in C^\infty(\Omega)$ such that $\boldsymbol{w}$ and $\nabla \times \boldsymbol{w}$ are not everywhere collinear.

Strategy of proof: the Clebsch representation $\boldsymbol{w} = \nabla\Psi \times \nabla\Theta$ reduces equation (1) to a single linear elliptic 2nd order PDE on each toroidal surface $\Psi = \textit{constant}$ for the unknown Θ in a periodic domain. Regular periodic solutions can be obtained by elliptic theory. A global solution Θ can then be constructed by smoothly joining solutions corresponding to different toroidal surfaces, thus providing a smooth solution $\boldsymbol{w}$ of (1) in a hollow toroidal volume Ω.

* Examples of smooth solutions foliated by toroidal surfaces that are not invariant under Euclidean isometries are constructed explicitly, and they are identified as anisotropic MHD equilibria.

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A Remark on Equation (1)

The main difficulty in (1) stems from $\nabla \cdot \boldsymbol{w} = 0$. Indeed, if this requirement is dropped, explicit solutions of $[(\nabla \times \boldsymbol{w}) \times \boldsymbol{w}] \times \nabla\Psi = \mathbf{0}$ can be obtained easily: the vector field

$$\boldsymbol{w} = f(\Psi,\alpha)\nabla\alpha + g(\Psi,\beta)\nabla\beta, \qquad \nabla\Psi\cdot\nabla\alpha = \nabla\Psi\cdot\nabla\beta = 0, \tag{11}$$

is a nontrivial solution of $(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = \lambda\nabla\Psi$ with

$$\lambda = -\frac{1}{2}\frac{\partial f^2}{\partial\Psi}|\nabla\alpha|^2 - \frac{1}{2}\frac{\partial g^2}{\partial\Psi}|\nabla\beta|^2 - \frac{\partial(fg)}{\partial\Psi}\nabla\alpha\cdot\nabla\beta. \tag{12}$$

Example: consider a family of toroidal surfaces corresponding to level sets of a function Ψ_ϵ defined by

$$\Psi_\epsilon = \Psi_0 + \frac{1}{2}\epsilon\sin(m\varphi), \qquad \Psi_0 = \frac{1}{2}[(r-r_0)^2 + z^2], \qquad m\in\mathbb{Z}, \qquad m\neq 0, \qquad \epsilon, r_0 > 0\,. \tag{13}$$

Here, (r,φ,z) denote cylindrical coordinates. It can be shown [arXiv:2211.10757] that $\mathcal{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\Psi_\epsilon = 0 \leftrightarrow \boldsymbol{a} = \boldsymbol{b} = \mathbf{0}$. Hence, the toroidal surfaces Ψ_ϵ are not invariant under continuous Euclidean isometries.

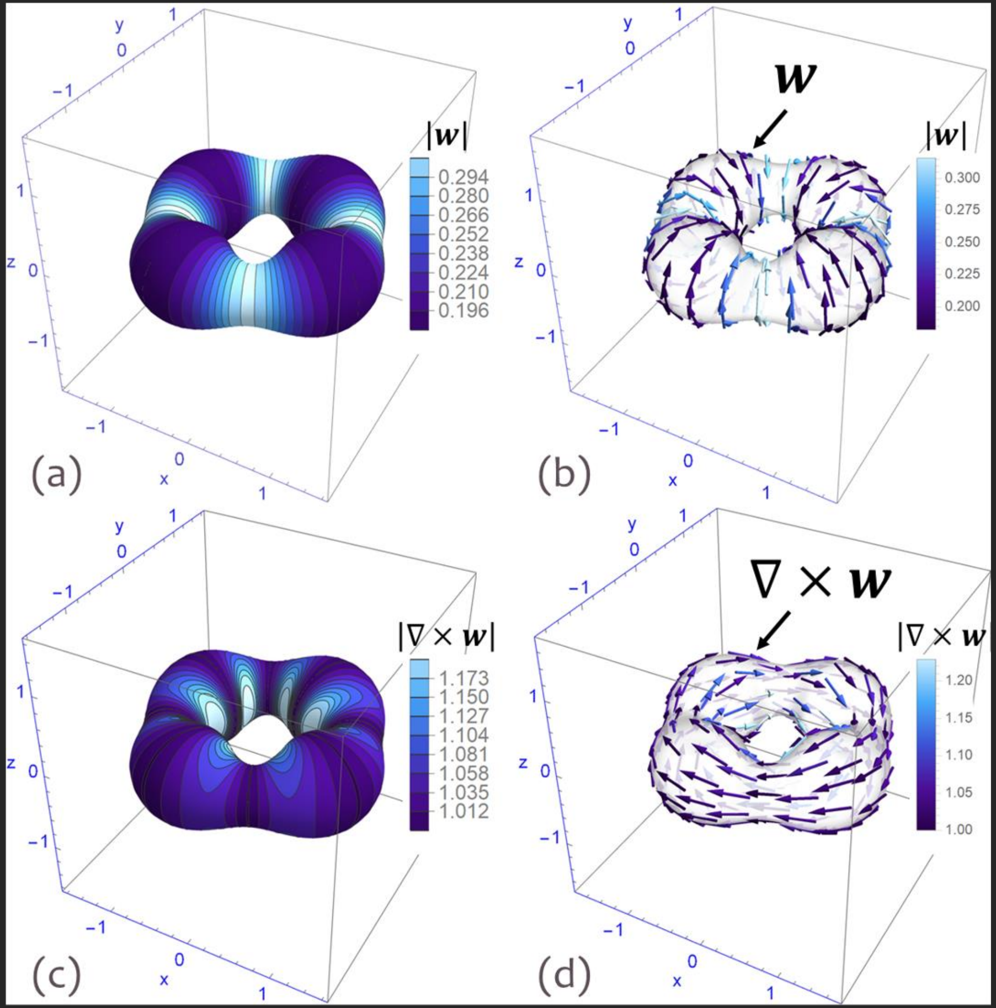
A Remark on Equation (1)

Let $\Omega \subset \mathbb{R}^3$ denote the volume enclosed by a contour of Ψ_ϵ and consider the vector field

$$\boldsymbol{w} = f(\Psi_\epsilon)\nabla\alpha, \qquad \alpha = \arctan\left(\frac{z}{r - r_0}\right). \tag{14}$$

It follows that

$$\begin{aligned}(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} &= -\frac{1}{2[2\Psi_\epsilon - \epsilon\sin(m\varphi)]}\frac{\partial f^2}{\partial \Psi_\epsilon}\nabla\Psi_\epsilon,\\ \nabla \cdot \boldsymbol{w} &= -\frac{zf(\Psi_\epsilon)}{r[2\Psi_\epsilon - \epsilon\sin(m\varphi)]}.\end{aligned} \tag{15}$$



The vector field $\boldsymbol{w}$ of (14) is shown with $\nabla \times \boldsymbol{w}$ on the contour $\Psi_\epsilon = 0.1$ in the figure on the right.

In the figure, $f = \Psi_\epsilon$, $r_0 = 1$, $\epsilon = 0.1$, and $m = 4$.

Reduction of Equation (1) Via Clebsch Potentials

For the time being, all quantities can be differentiated as many times as needed. Any solution of (1) with $(\nabla\times\boldsymbol{w})\times\boldsymbol{w}\neq\boldsymbol{0}$ satisfies $\boldsymbol{w}=\nabla\Psi\times\boldsymbol{q}$. Hence $\nabla\cdot\boldsymbol{w}=-\nabla\Psi\cdot\nabla\times\boldsymbol{q}=0$. In a small enough neighborhood, $\boldsymbol{q}=\nabla\vartheta$ for a single valued function ϑ (Lie-Barboux theorem [18-19]). Introducing a multivalued (angle) variable Θ, $\nabla\Theta\in\mathrm{Ker(curl)}$, we consider a candidate global solution

$$\boldsymbol{w}=\nabla\Psi\times\nabla\Theta\qquad \text{in }\ \Omega. \tag{16}$$

The functions Ψ and Θ are the Clebsch potentials of the Clebsch representation (16). Finding a solution of (1) in the form (16) amounts to determining Θ and $\boldsymbol{p}$ such that

$$\nabla\times(\nabla\Psi\times\nabla\Theta)=\nabla\Psi\times\boldsymbol{p}\qquad \text{in }\ \Omega. \tag{17}$$

Eq. (17) is equivalent to

$$\nabla\cdot[\nabla\Psi\times(\nabla\Theta\times\nabla\Psi)]=0\qquad \text{in }\ \Omega. \tag{18}$$

Solutions of (1) with representation (16) are solutions of (18). In the following we will study eq. (18).

[18] V. I. Arnold, in Mathematical Methods of Classical Mechanics, 2nd ed., Springer, New York, pp. 230-232 (1989). [19] M. de Le´on, in Methods of Differential Geometry in Analytical Mechanics, Elsevier, New York, pp. 250-253 (1989).

A 2nd Order Linear Degenerate Elliptic PDE

Eq. (18) can be written as

$$\sum_{i,j=1}^{3} |\nabla\Psi|^2 \left(\delta_{ij} - \frac{\Psi_i \Psi_j}{|\nabla\Psi|^2}\right) \Theta_{ij} + \sum_{i=1}^{3} \left(\frac{1}{2}|\nabla\Psi|_i^2 - \Psi_i \Delta\Psi\right) \Theta_i = 0 \qquad \text{in } \Omega. \tag{19}$$

The coefficient matrix

$$\mathfrak{a}_{ij} = |\nabla\Psi|^2 \left(\delta_{ij} - \frac{\Psi_i \Psi_j}{|\nabla\Psi|^2}\right), \qquad i,j = 1,2,3. \tag{20}$$

Is symmetric and positive semi-definite

$$\mathfrak{a}_{ij}\xi^i\xi^j = |\nabla\Psi \times \boldsymbol{\xi}|^2 \geq 0, \qquad \boldsymbol{\xi} \in \mathbb{R}^3, \qquad \boldsymbol{x} \in \Omega. \tag{21}$$

- For given Ψ, eq. (18) is a 2nd order linear degenerate elliptic PDE for the unknown Θ.
- If Θ is a solution of (18), so if $\Theta + f(\Psi)$.

Variational Formulation of Equation (18)

Consider the magnetic energy (kinetic energy in the fluid analogy)

$$E_\Omega = \frac{1}{2}\int_\Omega \boldsymbol{w}^2 dV = \frac{1}{2}\int_\Omega |\nabla\Psi \times \nabla\Theta|^2\, dV\,. \tag{22}$$

If variations $\delta\Theta$ vanish on $\partial\Omega$,

$$\begin{aligned} \delta E_\Omega &= -\int_\Omega \delta\Theta\nabla\cdot[\nabla\Psi\times(\nabla\Theta\times\nabla\Psi)]\, dV \\ \rightarrow \quad & \nabla\cdot[\nabla\Psi\times(\nabla\Theta\times\nabla\Psi)] = 0 \qquad \text{in } \Omega. \end{aligned} \tag{23}$$

- Hence, stationary points of E_Ω correspond to solutions of (18). However, the functional E_Ω lacks coercivity $E_\Omega \geq c||\Theta||^2$ with respect to standard norms due to the cross product.
- The degeneracy is not expected to prevent the existence of solutions, but simply to affect their uniqueness.

Reformulation as a 2nd Order Linear Elliptic PDE on a Toroidal Surface

A global solution of (1) can be obtained by solving (18) on each toroidal surface $\Sigma_{\Psi_0} = \{\boldsymbol{x} \in \Omega; \Psi(\boldsymbol{x}) = \Psi_0 \in \mathbb{R}\}$ and by patching solutions. The degeneracy of eq. (18) can be removed by fixing the mean value $\langle\Theta\rangle$ of the unknown Θ over each surface.

Consider curvilinear coordinates $(x^1, x^2, x^3) = (\mu, \nu, \Psi)$ with $\mu, \nu \in [0,2\pi)$ angle coordinates spanning the toroidal surfaces Σ_{Ψ_0}, ∂_i, $i = 1,2,3$, tangent vectors, $J = \nabla\mu \cdot \nabla\nu \times \nabla\Psi$ the Jacobian determinant, and $g_{ij} = \partial_i \cdot \partial_j$ the covariant metric tensor. On each Σ_{Ψ_0} eq. (18) becomes

$$\frac{\partial}{\partial\mu}\left[J\left(g_{\nu\nu}\frac{\partial\Theta}{\partial\mu} - g_{\mu\nu}\frac{\partial\Theta}{\partial\nu}\right)\right] + \frac{\partial}{\partial\nu}\left[J\left(g_{\mu\mu}\frac{\partial\Theta}{\partial\nu} - g_{\mu\nu}\frac{\partial\Theta}{\partial\mu}\right)\right] = 0 \qquad \text{in } \Sigma_{\Psi_0}. \tag{24}$$

Here $g_{11} = g_{\mu\mu}$, $g_{12} = g_{\mu\nu}$, $g_{22} = g_{\nu\nu}$. Eq. (24) is equivalent to:

$$\begin{aligned} g_{\nu\nu}\Theta_{\mu\mu} - 2g_{\mu\nu}\Theta_{\mu\nu} + g_{\mu\mu}\Theta_{\nu\nu} + \left[\frac{J_\mu}{J}g_{\nu\nu} + \frac{\partial g_{\nu\nu}}{\partial\mu} - \frac{J_\nu}{J}g_{\mu\nu} - \frac{\partial g_{\mu\nu}}{\partial\nu}\right]\Theta_\mu \\ + \left[\frac{J_\nu}{J}g_{\mu\mu} + \frac{\partial g_{\mu\mu}}{\partial\nu} - \frac{J_\mu}{J}g_{\mu\nu} - \frac{\partial g_{\mu\nu}}{\partial\mu}\right]\Theta_\nu = 0 \qquad \text{in } \Sigma_{\Psi_0}. \end{aligned} \tag{25}$$

Reformulation as a 2nd Order Linear Elliptic PDE on a Toroidal Surface

Eq. (25) has the form

$$\sum_{i,j=1}^{2} a_{ij}\Theta_{ij} + \text{lower order terms} = 0 \qquad \text{in } \Sigma_{\Psi_0}, \tag{26}$$

where the coefficient matrix A with components a_{ij}, $i,j = 1,2$, is given by

$$A = A^T = \begin{bmatrix} g_{\nu\nu} & -g_{\mu\nu} \\ -g_{\mu\nu} & g_{\mu\mu} \end{bmatrix}. \tag{27}$$

Both eigenvalues $\lambda_\pm = \mathrm{Tr}A \pm \sqrt{(\mathrm{Tr}A)^2 - 4\det A}$ are real and positive with $\lambda_+ \geq \lambda_- > 0$ since $\mathrm{Tr}A = g_{\mu\mu} + g_{\nu\nu} > 0$, $\det A = g_{\mu\mu}g_{\nu\nu} - g_{\mu\nu}^2 = \left|\partial_\mu \times \partial_\nu\right|^2 > 0$, and $(\mathrm{Tr}A)^2 > (\mathrm{Tr}A)^2 - 4\det A = \left(g_{\mu\mu} - g_{\nu\nu}\right)^2 + 4g_{\mu\nu}^2 \geq 0$.

Hence, eq. (24) is a 2nd order linear (strictly) elliptic PDE on each toroidal surface Σ_{Ψ_0}:

$$a_{ij}\xi^i\xi^j \geq \lambda_-|\boldsymbol{\xi}|^2 \geq 0, \qquad \boldsymbol{\xi} \in \mathbb{R}^2, \qquad (\mu,\nu) \in [0,2\pi), \qquad \Psi = \Psi_0. \tag{28}$$

Boundary Conditions

Under appropriate boundary conditions, solutions Θ of (24) exist and are unique by elliptic theory [20]. However, not all boundary conditions result in a nontrivial $\boldsymbol{w}$. For example, setting $\Theta = 0$ on ∂D, with $D = (0,2\pi)^2$, gives $\Theta = 0$ in D, and thus $\boldsymbol{w} = \nabla\Psi \times \nabla\Theta = \boldsymbol{0}$ in D. Furthermore, even if $\nabla\Theta \neq \boldsymbol{0}$, there is no guarantee that $\nabla\Theta$, and thus $\boldsymbol{w}$, is periodic in μ and ν. We therefore look for solutions of the type:

$$\Theta = \mu + \rho, \qquad \langle\rho\rangle = \int_D d\mu d\nu\, \rho = 0, \qquad \rho \text{ periodic in } D \text{ (with periodic derivatives)} \tag{29}$$

Substituting eq. (29), eq. (24) becomes:

$$\frac{\partial}{\partial\mu}\left[J\left(g_{\nu\nu}\frac{\partial\rho}{\partial\mu} - g_{\mu\nu}\frac{\partial\rho}{\partial\nu}\right)\right] + \frac{\partial}{\partial\nu}\left[J\left(g_{\mu\mu}\frac{\partial\rho}{\partial\nu} - g_{\mu\nu}\frac{\partial\rho}{\partial\mu}\right)\right] = \frac{\partial}{\partial\nu}(Jg_{\mu\nu}) - \frac{\partial}{\partial\mu}(Jg_{\nu\nu}) \quad \text{in } D. \tag{30}$$

Eq. (30) is strictly elliptic (it has the same coefficient matrix A). If a solution ρ of (30) exists, the corresponding Θ is nontrivial since $\nabla\Theta = \nabla\mu + \nabla\rho$ is periodic in μ and ν and $\langle\Theta_\mu\rangle = \langle 1 + \rho_\mu\rangle = 4\pi^2 \rightarrow w^\nu = J\Theta_\mu \neq 0$.

- Eq. (1) has been reduced to the existence of a periodic solution (with periodic derivatives) of (30) that depends in a regular fashion on the surface label Ψ.

[20] D. Gilbarg and N. S. Trudinger, in Elliptic Partial Differential Equations of Second Order, Springer, p. 107 (1998).

Boundary Conditions

Although the coefficients in (30) are periodic in μ and ν, Dirichlet boundary conditions for ρ on ∂D are not enough to ensure the periodicity of ρ_μ, ρ_ν, ρ_Ψ and so on: the regularity of the solution $\boldsymbol{w} = \nabla\Psi \times \nabla\Theta$ will reflect the degree of periodicity of ρ and its partial derivatives. Indeed,

$$\boldsymbol{w} = J\left(\frac{\partial\Theta}{\partial\mu}\partial_\nu - \frac{\partial\Theta}{\partial\nu}\partial_\mu\right),$$

$$\begin{aligned}\nabla\times\boldsymbol{w} = J&\left\{\frac{\partial}{\partial\Psi}\left[J\left(g_{\mu\nu}\frac{\partial\Theta}{\partial\mu} - g_{\mu\mu}\frac{\partial\Theta}{\partial\nu}\right)\right] - \frac{\partial}{\partial\mu}\left[J\left(g_{\nu\Psi}\frac{\partial\Theta}{\partial\mu} - g_{\Psi\mu}\frac{\partial\Theta}{\partial\nu}\right)\right]\right\}\partial_\nu \\ +J&\left\{\frac{\partial}{\partial\Psi}\left[J\left(g_{\mu\nu}\frac{\partial\Theta}{\partial\nu} - g_{\nu\nu}\frac{\partial\Theta}{\partial\mu}\right)\right] + \frac{\partial}{\partial\nu}\left[J\left(g_{\nu\Psi}\frac{\partial\Theta}{\partial\mu} - g_{\Psi\mu}\frac{\partial\Theta}{\partial\nu}\right)\right]\right\}\partial_\mu\end{aligned} \tag{31}$$

For $\boldsymbol{w}$ and $\nabla\times\boldsymbol{w}$ to be continuous in Ω, $\Theta_\mu = 1 + \rho_\mu$, $\Theta_\nu = \rho_\nu$, $\Theta_{\mu\mu} = \rho_{\mu\mu}$, $\Theta_{\mu\nu} = \rho_{\mu\nu}$ and $\Theta_{\nu\nu} = \rho_{\nu\nu}$ must be periodic in μ and ν. Conversely, if they fail to be periodic, $\boldsymbol{w}$ and $\nabla\times\boldsymbol{w}$ will exhibit discontinuities on each toroidal surface in correspondence of the curves $\gamma_{\partial D} = \{\boldsymbol{x} \in \Omega : (\mu,\nu) \in \partial D, \Psi = \Psi_0\}$.

Existence of Smooth Unique Periodic Solution ρ of Equation (30)

Strategy: since there are no requirements on the boundary values of $\Theta = \mu + \rho$ on ∂D, the idea is to construct a weak periodic solution of equation (30) in a two-dimensional lattice extending over $\mathbb{R}^2$ with unit cell D by introducing an appropriate Hilbert space $H^1_{\mathrm{per}}(D) \subset H^1(D)$ containing periodic functions. Then, interior regularity can be used to infer smoothness of weak solutions, and thus periodicity of their derivatives.

Lemma 1. *Let $V = D \times U$ denote a doubly periodic three-dimensional domain spanned by coordinates $\mu, \nu \in [0,2\pi)$, $\Psi \in U$, with $D = (0,2\pi)^2$ and $U \subset \mathbb{R}$ a bounded open interval. Define $(x^1, x^2) = (\mu, \nu)$. Let $\alpha^{ij} \in C^\infty(\mathbb{R}^2 \times U)$, $i, j = 1,2$, and $S \in C^\infty(\mathbb{R}^2 \times U)$ be smooth functions which are periodic in D. Further assume that $\langle S \rangle = \int_D d\mu d\nu\, S = 0$, and that α^{ij} is strictly elliptic on each level set of Ψ, i.e.*

$$\alpha^{ij}\xi_i\xi_j \geq \lambda|\boldsymbol{\xi}|^2, \qquad \boldsymbol{\xi} \in \mathbb{R}^2, \qquad \mu, \nu \in [0,2\pi), \qquad \Psi \in U,$$

for some positive constant λ. Then, the boundary value problem

$$\frac{\partial}{\partial x^i}\left(\alpha^{ij}\frac{\partial \rho}{\partial x^j}\right) = S, \qquad \langle \rho \rangle = \int_0^{2\pi} d\mu \int_0^{2\pi} d\nu\ \rho = 0 \qquad \text{in } V, \qquad \rho \text{ periodic in } D,$$

admits a unique periodic solution $\rho \in C^\infty(\mathbb{R}^2 \times U)$ with periodic derivatives of all orders. In particular, for fixed $\Psi \in U$ the function of two variables $\rho^\Psi = \rho(\mu, \nu, \Psi)$ satisfies $\rho^\Psi \in C^\infty(\mathbb{R}^2) \cap H^1_{\mathrm{per}}(D)$. Here,

$$H^1_{\mathrm{per}}(D) = \{\rho^\Psi \in H^1(D);\ \langle \rho^\Psi \rangle = 0, \rho^\Psi \text{ periodic in } D\}.$$

Proof of Lemma 1

Proof: in a two-dimensional lattice with unit cell D, a periodic solution satisfies

$$\rho(\mu,\nu,\Psi) = \rho(\mu+2\pi m,\nu+2\pi n,\Psi), \qquad m,n \in \mathbb{Z}. \tag{32}$$

Hence, If derivatives of ρ exist, they are periodic as well.

Next, note that for each $\Psi \in U$ the strict ellipticity of α^{ij}, the regularity and periodicity of α^{ij} and S, and the condition $\langle S\rangle = 0$ guarantee that the boundary value problem (33) admits a unique solution $\rho^\Psi \in H^1_{\mathrm{per}}(D)$ [21].

$$\frac{\partial}{\partial x^i}\left(\alpha^{ij}\frac{\partial\rho^\Psi}{\partial x^j}\right) = S, \qquad \langle\rho^\Psi\rangle = 0 \qquad \text{in } D, \qquad \rho^\Psi \text{ periodic in } D. \tag{33}$$

Indeed, note that $H^1_{\mathrm{per}}(D)$ can be identified with the completion of $C^\infty_{\mathrm{per}}(D)$ with respect to the H^1 norm, with

$$C^\infty_{\mathrm{per}}(D) = C^\infty(\mathbb{R}^2) \cap H^1_{\mathrm{per}}(D) = \{\rho^\Psi \in C^\infty(\mathbb{R}^2);\ \langle\rho^\Psi\rangle = 0, \rho^\Psi \text{ periodic in } D\}. \tag{34}$$

[21] A. Bensoussan, J. L. Lions, and G. Papanicolau, Elliptic operators, in Asymptotic analysis for periodic structures, North-Holland, pp. 13-14 (1978).

Proof of Lemma 1

On the other hand, the weak formulation of (33) is

$$(\rho^{\Psi},\psi)+\mathcal{F}_S[\psi]=\int_D\left(\alpha^{ij}\frac{\partial\psi}{\partial x^i}\frac{\partial\rho^{\Psi}}{\partial x^j}+S\psi\right)d\mu d\nu=0\qquad\forall\psi\in H^1_{\mathrm{per}}(D),\tag{35}$$

where the inner product

$$(\rho^{\Psi},\psi)=\int_D\alpha^{ij}\frac{\partial\psi}{\partial x^i}\frac{\partial\rho^{\Psi}}{\partial x^j}d\mu d\nu\,,\tag{36}$$

defines a norm $\|\rho^{\Psi}\|_{H^1_{\mathrm{per}}(D)}=(\rho^{\Psi},\rho^{\Psi})^{1/2}$ in $H^1_{\mathrm{per}}(D)$ due to the strict ellipticity of α^{ij}. Indeed, using the Poincaré inequality [22], we have

$$(\rho^{\Psi},\rho^{\Psi})\geq\lambda\left\|\nabla_{(\mu,\nu)}\rho^{\Psi}\right\|^2_{L^2(D)}\geq C\|\rho^{\Psi}\|^2_{H^1(D)},\tag{37}$$

Hence, $H^1_{\mathrm{per}}(D)$ is a Hilbert space with respect to the norm $\|\cdot\|_{H^1_{\mathrm{per}}(D)}$.

[22] J. Necas, The Poincaŕe inequality, in Direct Methods in the Theory of Elliptic Operators, Spinger, p. 7 (2012).

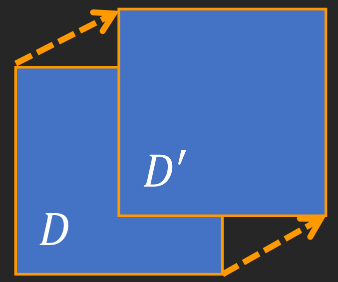
Proof of Lemma 1

Next, the linear functional

$$\mathcal{F}_S[\psi] = \int_D S\psi \, d\mu dv \leq C\|\psi\|_{H^1_{\mathrm{per}}(D)}, \tag{35}$$

Is bounded. Hence, the Riesz representation theorem guarantees the existence of a unique element $\rho^{\Psi} \in H^1_{\mathrm{per}}(D)$ such that $\mathcal{F}_S\,[\psi] = -(\rho^{\Psi}, \psi)$, which thus provides a weak solution of (33).

The construction holds even if the origin of the cell D is shifted by an arbitrary amount in $\mathbb{R}^2$. Let $D' \subset \mathbb{R}^2$ denote the shifted cell and $\rho'^{\Psi} \in H^1_{\mathrm{per}}(D') = H^1_{\mathrm{per}}(D)$ the corresponding solution. By interior regularity, any irregularity of the solution ρ'^{Ψ} that may occur on the boundary $\partial D'$ cannot affect the interior of the domain, and it can be shown that the regularity of α^{ij} and S is propagated to ρ'^{Ψ}. In particular, $\rho'^{\Psi} \in C^{\infty}(D')$ (see [23]). Since we may take $D' \cap D \neq \emptyset$ and $\rho^{\Psi} = \rho'^{\Psi}$ by uniqueness, this also implies the regularity of the derivatives of ρ^{Ψ} at the original cell boundary ∂D, and thus their periodicity. We conclude that $\rho^{\Psi} \in C^{\infty}_{\mathrm{per}}(D)$ and that all partial derivatives of any order of the function ρ^{Ψ} are periodic functions in D.



[23] L. C. Evans, Second-order elliptic equations, in Partial Differential Euations, 2nd ed., American Mathematical Society, p. 336 and p. 342 (2010).

Proof of Lemma 1

We are left with the task of showing that solutions ρ^Ψ of (33) define a smooth function ρ of Ψ in U. Define

$$L = \frac{\partial}{\partial x^i}\left(\alpha^{ij}\frac{\partial}{\partial x^j}\right), \qquad L_\Psi = \frac{\partial}{\partial x^i}\left(\alpha^{ij}_\Psi\frac{\partial}{\partial x^j}\right) \tag{35}$$

The first eq. in (33) is $L\rho^\Psi = S$. Furthermore, the linear operator L defines an invertible linear mapping from $C^\infty_{\mathrm{per}}(D)$ to itself. It follows that

$$0 = \frac{\partial(LL^{-1})}{\partial\Psi} = L_\Psi L^{-1} + LL^{-1}_\Psi. \tag{36}$$

Since $L^{-1}(0) = 0$, application of L^{-1} to (36) gives

$$L^{-1}_\Psi = -L^{-1}L_\Psi L^{-1} \quad \rightarrow \quad \frac{\partial\rho}{\partial\Psi} = \frac{\partial(L^{-1}S)}{\partial\Psi} = L^{-1}(S_\Psi - L_\Psi\rho) \quad \rightarrow \quad \left(\frac{\partial\rho}{\partial\Psi}\right)^\Psi \in C^\infty_{\mathrm{per}}(D). \tag{37}$$

Higher order derivatives of L^{-1} and ρ can be evaluated by differentiating (36) and $\rho = L^{-1}S$. Hence, for each $\Psi \in U$ derivatives of ρ with respect to Ψ of all orders exist and belong to $C^\infty_{\mathrm{per}}(D)$. It follows that ρ is smooth in Ψ, and therefore provides a unique solution $\rho \in C^\infty(\mathbb{R}^2 \times U)$ with $\rho^\Psi \in C^\infty(\mathbb{R}^2) \cap H^1_{\mathrm{per}}(D)$. ■

Proof of Theorem 1

Proof: the hypothesis of lemma 1 are satisfied with $\alpha^{ij} = Ja^{ij}$ and source term

$$S = \frac{\partial}{\partial \nu}\left(J g_{\mu\nu}\right) - \frac{\partial}{\partial \mu}(J g_{\nu\nu}). \tag{38}$$

Let $\rho \in C^{\infty}(\mathbb{R}^2 \times U)$ denote the periodic classical solution of eq. (30) obtained from lemma 1. Evidently, $\rho \in C^{\infty}(\Omega)$ as well. Setting $\Theta = \mu + \rho$, it follows that the vector field

$$\boldsymbol{w} = \nabla\Psi \times \nabla\Theta = J\left(\Theta_{\mu}\partial_{\nu} - \Theta_{\nu}\partial_{\mu}\right) = J\partial_{\nu} + \nabla\Psi \times \nabla\rho, \tag{39}$$

is a solution $\boldsymbol{w} \in C^{\infty}(\Omega)$ of (1). Furthermore, $\boldsymbol{w} \neq \boldsymbol{0}$ since $\langle\Theta_{\mu}\rangle = 4\pi^2$. It may happen however that the solution $\boldsymbol{w}$ is a curl-free (vacuum) solution $\nabla \times \boldsymbol{w} = \boldsymbol{0}$, or a Beltrami field $\nabla \times \boldsymbol{w} = \hat{h}\,\boldsymbol{w}$ for some proportionality coefficient $\hat{h}(\boldsymbol{x})$. Nevertheless, denoting with $f(\Psi) \neq 0$ any smooth function of the variable Ψ such that $\frac{\partial f}{\partial \Psi} \neq 0$, it follows that the vector field $\boldsymbol{w}' = f(\Psi)\boldsymbol{w}$ is a nontrivial solution of (1). Indeed,

$$(\nabla \times \boldsymbol{w}') \times \boldsymbol{w}' = -\frac{1}{2}\frac{\partial f^2}{\partial \Psi}\boldsymbol{w}^2\nabla\Psi \neq \boldsymbol{0}, \qquad \nabla \cdot \boldsymbol{w}' = \frac{\partial f}{\partial \Psi}\nabla\Psi \cdot \boldsymbol{w} = 0. \quad \blacksquare \tag{40}$$

Remarks

Remark 1: no symmetry requirements on $\boldsymbol{w}$ and Ψ in theorem 1.

Remark 2: In the original formulation of the problem (1), the domain Ω is a torus. However, the result of theorem 1 applies to a hollow torus. For the solution $\boldsymbol{w}$ of theorem 1 to hold in the hollow region as well, the vector field $\boldsymbol{w}$ must be well defined when approaching the toroidal axis. This is often the case (see later examples).

Remark 3: in the study of the vorticity equation for fluid flows over two-dimensional surfaces parametrized by Ψ and embedded in three-dimensional Euclidean space,

$$\frac{\partial \omega^{\Psi}}{\partial t} = J[\Theta, \omega^{\Psi}], \qquad [f, g] = f_{\mu} g_{\nu} - f_{\nu} g_{\mu},$$

the relation between the component of the vorticity $\omega^{\Psi} = \boldsymbol{\omega} \cdot \nabla\Psi$ and the stream function Θ is precisely [24]

$$\nabla \cdot [\nabla\Psi \times (\nabla\Theta \times \nabla\Psi)] = -\omega^{\Psi}.$$

The result of lemma 1 thus implies that one can solve for the stream function Θ knowing the vorticity ω^{Ψ}. Notice that the topology of the level sets of Ψ does not need to be toroidal.

[24] N. Sato and M. Yamada, Vorticity equation on surfaces with arbitrary topology embedded in three-dimensional Euclidean space, J. Math. Phys. **63**, 093101 (2022).

Example of Numerical Solution with Discontinuity

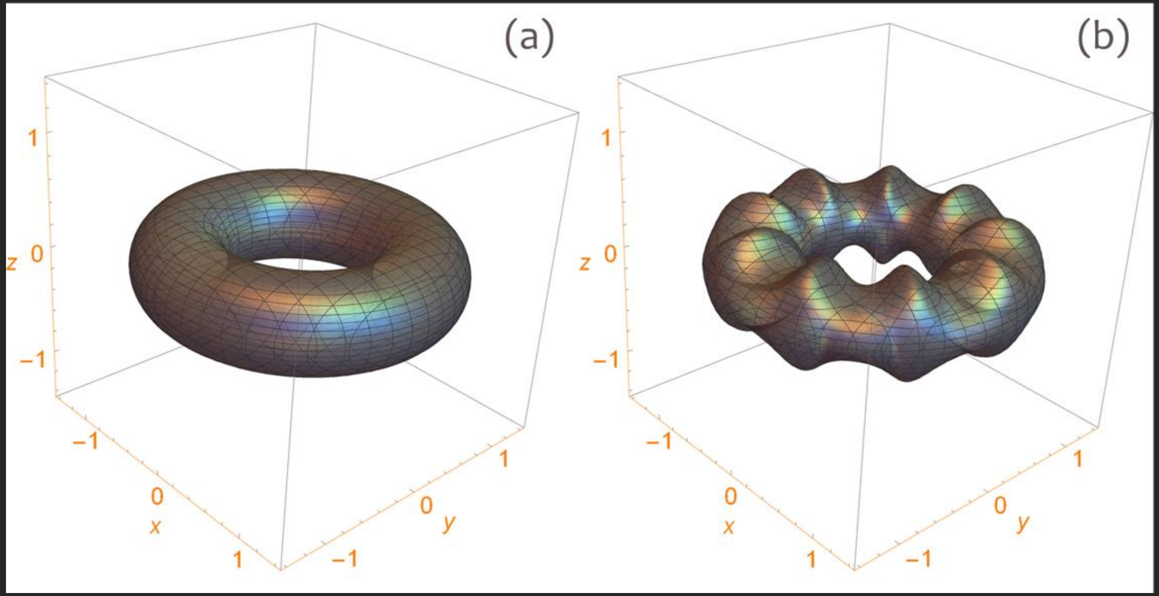
The numerical example below clarifies the role played by periodic boundary conditions in ensuring the regularity of the solution $\boldsymbol{w}$ of (1) and its derivatives. Consider the family of toroidal surfaces

$$\Psi = \frac{1}{2}(r - r_0)^2 + \frac{1}{2}\mathcal{E}[z - h(\varphi, z)]^2, \qquad r_0, \mathcal{E} \in \mathbb{R}_{>0} \tag{41}$$

A suitable choice of h produces tori without continuous Euclidean isometries. For example, $h = \epsilon z \sin(m\varphi)$ with $\epsilon \in \mathbb{R}_{>0}$ and integer $m \neq 0$ gives $\mathcal{L}_{\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{x}}\Psi = 0 \quad \leftrightarrow \quad \boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}.$

Define curvilinear coordinates $(\mu, \nu, \Psi) = (\varphi, \vartheta, \Psi)$ with $\vartheta = \arctan\left(\frac{z}{r - r_0}\right)$ and consider equation (30) with Dirichlet boundary conditions $\rho = 0$ on ∂D.

The solution $\rho \in C^{1,\alpha}(D)$, $0 < \alpha < 1$, will be periodic in D, although only the partial derivative of ρ tangential to the boundary ∂D will be periodic, while the normal component will not [25-26].

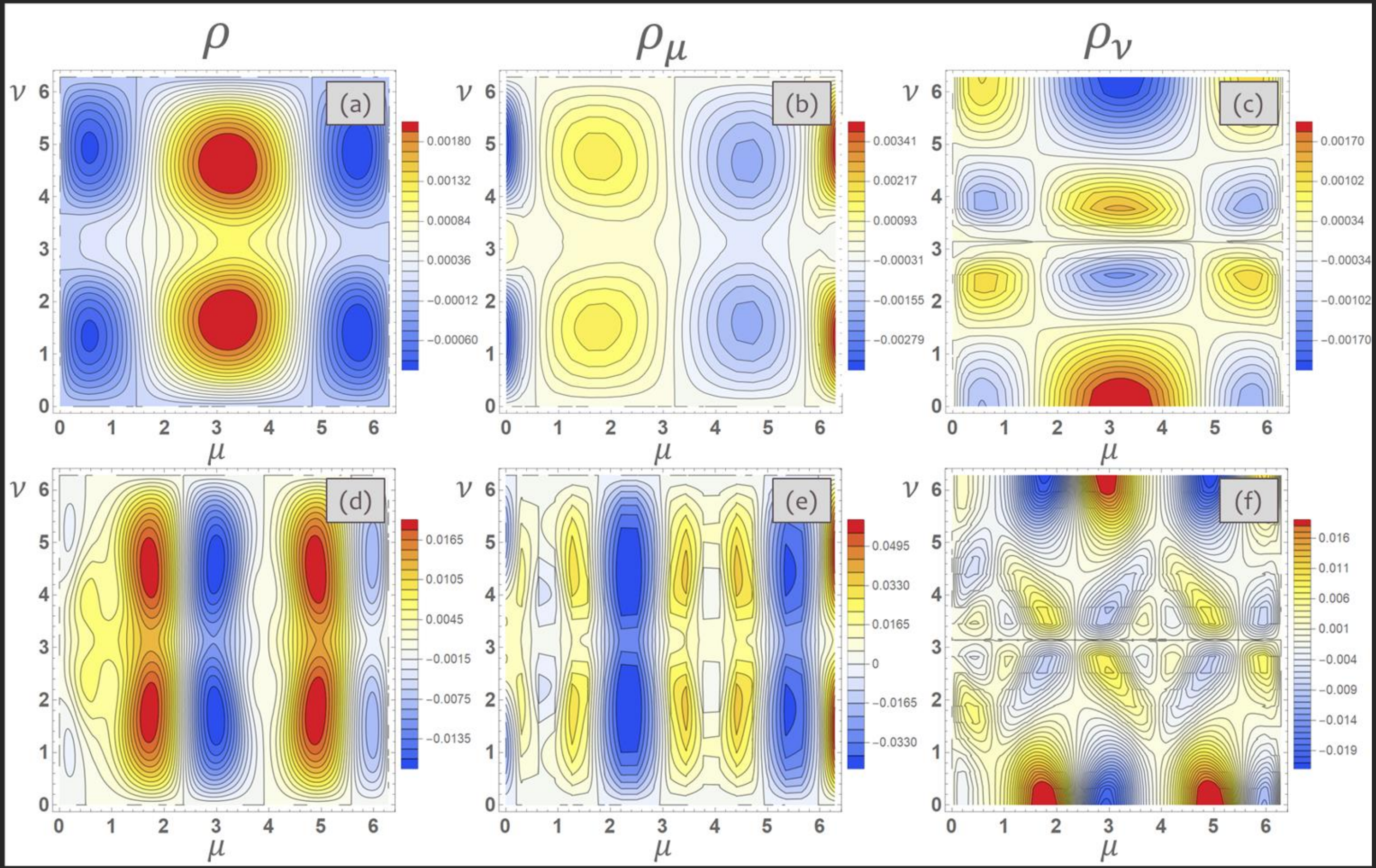


(a) Axially symmetric torus $\Psi = 0.08$ with $r_0 = \mathcal{E} = 1$, $h = 0$.
(b) Torus without continuous Euclidean isometries $\Psi = 0.08$ with $r_0 = 1$, $\mathcal{E} = 1.6$, $h = 0.3\, z\, \sin(9\varphi)$.

[25] A. Azzam, SIAM J. Math. Anal. **11**, 2 (1980). [26] A. Azzam, SIAM J. Math. Anal. **12**, 2 (1981)..

Example of Numerical Solution with Discontinuity

Note that ρ is periodic in D, but $\rho_\mu = \Theta_\mu - 1$ is periodic only between $\nu = 0$ and $\nu = 2\pi$, and $\rho_\nu = \Theta_\nu$ between $\mu = 0$ and $\mu = 2\pi$. Hence, $\boldsymbol{w} = \nabla\Psi \times \nabla\Theta = J\left(\Theta_\mu \partial_\nu - \Theta_\nu \partial_\mu\right)$ exhibits discontinuities on ∂D.



First row: $\Psi = 0.16$ with $r_0 = 1$, $\mathcal{E} = 1.6$, $h = 0.03\, z\, \sin(\varphi)$.
Second row: $\Psi = 0.08$ with $r_0 = 1$, $\mathcal{E} = 1.6$, $h = 0.3\, z\, \sin(2\varphi)$.

Example of Smooth Solution and Relation with Anisotropic MHD

In the following, we construct examples of smooth solutions $\boldsymbol{w} \in C^{\infty}(\Omega)$ of eq. (1) such that $(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} \neq \boldsymbol{0}$ in a toroidal domain Ω. To this end, the observation below is useful

Proposition 1. *Let $\Omega \subset \mathbb{R}^3$ be a toroidal volume with boundary $\partial\Omega$ foliated by toroidal surfaces corresponding to level sets of a function $\Psi \in C^1(\bar{\Omega})$. Let $\boldsymbol{\xi} \in L^2_H(\Omega)$ be a harmonic vector field in Ω, with*

$$L^2_H(\Omega) = \{\boldsymbol{\xi} \in L^2(\Omega); \nabla \times \boldsymbol{\xi} = \boldsymbol{0}, \nabla \cdot \boldsymbol{\xi} = 0, \boldsymbol{\xi} \cdot \boldsymbol{n} = 0\},$$

where $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$. Further assume that $\boldsymbol{\xi}$ is foliated by Ψ, that is

$$\boldsymbol{\xi} \cdot \nabla\Psi = 0 \quad \text{in } \Omega.$$

Then, the vector field $\boldsymbol{w} \in H^1_{\sigma\sigma}(\Omega)$ defined as

$$\boldsymbol{w} = f(\Psi)\boldsymbol{\xi},$$

where f is any $C^1(\bar{\Omega})$ function of Ψ, is a nontrivial solution of (1) in Ω such that

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = -\frac{1}{2}\frac{\partial f^2}{\partial \Psi}|\boldsymbol{\xi}|^2\nabla\Psi, \qquad \nabla \cdot \boldsymbol{w} = 0.$$

Here,

$$H^1_{\sigma\sigma}(\Omega) = \{\boldsymbol{w} \in L^2_\sigma(\Omega); \nabla \times \boldsymbol{w} \in L^2_\sigma(\Omega)\}, \qquad L^2_\sigma(\Omega) = \{\boldsymbol{w} \in L^2(\Omega); \nabla \cdot \boldsymbol{w} = 0, \boldsymbol{w} \cdot \boldsymbol{n} = 0\}.$$

Recall that the dimension of the linear space $L^2_H(\Omega)$ is given by the genus of $\partial\Omega$. For a toroidal surface with genus 1 the space of harmonic vector fields $\boldsymbol{\xi} \in L^2_H(\Omega)$ is therefore 1-dimensional [27-28].

[27] G. Schwarz, in Hodge decomposition - a method for solving boundary value problems, Springer, pp. 67-72 (1995). [28] D. Pfefferľe, L. Noakes, and D. Perrella, Gauge freedom in magnetostatics and the effect on helicity in toroidal volumes, J. Math. Phys. **62**, 093505 (2021).

Example of Smooth Solution and Relation with Anisotropic MHD

Proposition 1 suggests that solutions of (1) can be obtained by identifying harmonic vector fields foliated by toroidal surfaces. Example: $\xi_0 = \nabla\varphi$ is foliated by axially symmetric tori $\Psi_0 = \frac{1}{2}[(r - r_0)^2 + z^2]$. To break axial symmetry, we perturb the toroidal angle:

$$\eta = \varphi + \epsilon\sigma, \qquad \epsilon \in \mathbb{R}_{>0}, \qquad \Delta\sigma = 0 \quad \to \quad \xi_\epsilon = \nabla\eta \in L^2_H(\Omega), \tag{42}$$

where the toroidal domain Ω is bounded by a contour of a function Ψ_ϵ satisfying $\boldsymbol{\xi}_\epsilon \cdot \nabla\Psi_\epsilon = 0$. Choosing $\sigma = x$,

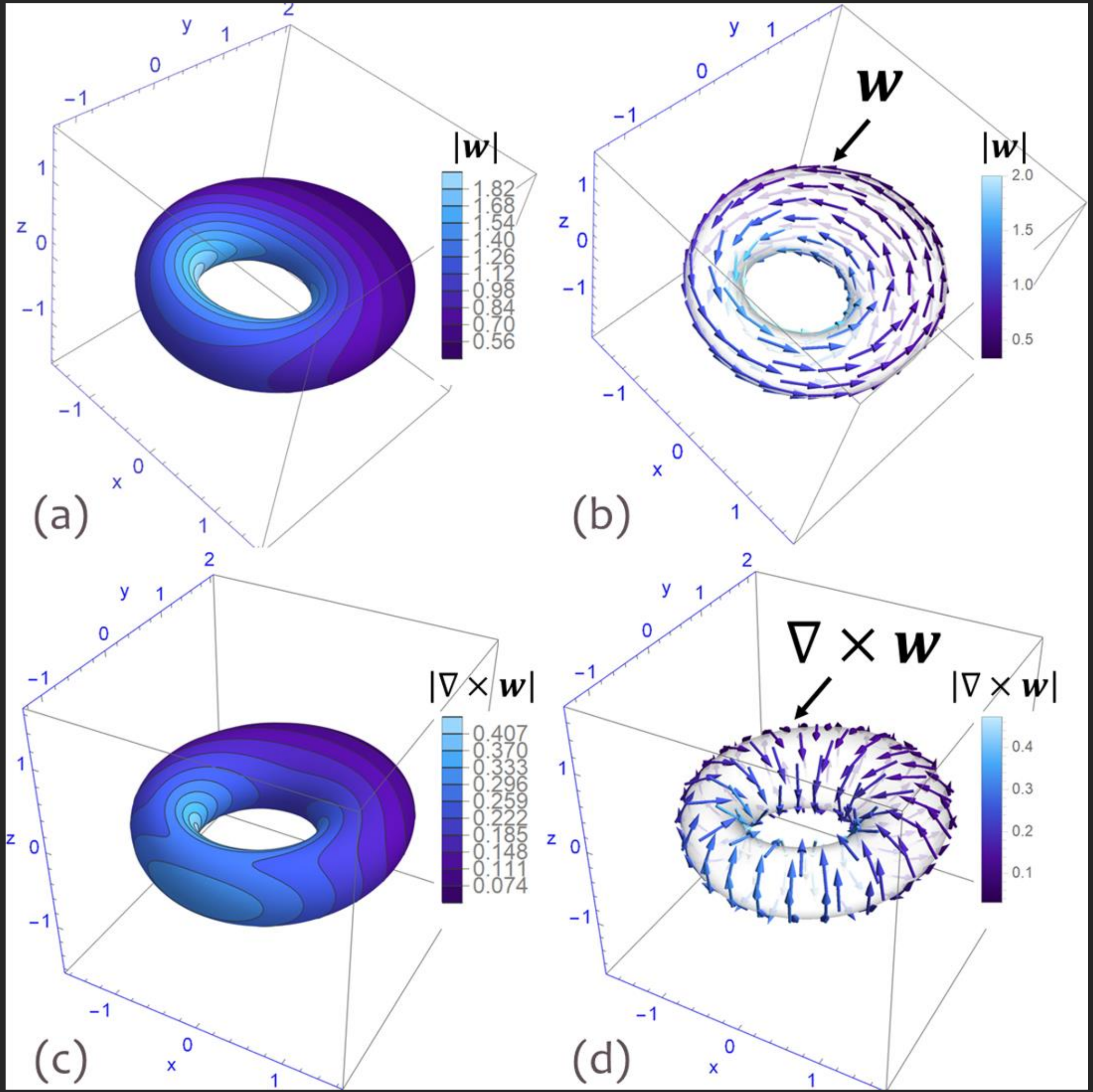
$$\Psi_\epsilon = \frac{1}{2}[(re^{-\epsilon y} - r_0)^2 + z^2] \quad \to \quad \boldsymbol{w} = f(\Psi_\epsilon)\boldsymbol{\xi}_\epsilon. \tag{43}$$

Note that $\boldsymbol{w}$ is smooth within Ω for a suitable f. In addition,

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = -\frac{1}{2}\frac{\partial f^2}{\partial\Psi_\epsilon}\frac{(1-\epsilon y)^2 + \epsilon^2 x^2}{r^2}\nabla\Psi, \qquad \nabla \cdot \boldsymbol{w} = 0. \tag{44}$$

Furthermore, it can be shown that both $\boldsymbol{w}$ and Ψ_ϵ are not invariant under continuous Euclidean isometries.

Example of Smooth Solution and Relation with Anisotropic MHD



Here, $\Psi_\epsilon = 0.08$, $r_0 = 1$, $\epsilon = 0.18$, $f = \mathrm{e}^{\Psi_\epsilon/2}$, $\sigma = x$.

Example of Smooth Solution and Relation with Anisotropic MHD

The vector field (43) can be regarded as a steady solution of anisotropic MHD [29-30],

$$(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = \nabla \cdot \Pi, \qquad \nabla \cdot \boldsymbol{w} = 0 \qquad \text{in } \ \Omega, \qquad \Pi^{ij} = \left(P - \frac{1}{2}\gamma \boldsymbol{w}^2\right)\delta^{ij} + \gamma w^i w^j. \tag{45}$$

Indeed,

$$(1-\gamma)(\nabla \times \boldsymbol{w}) \times \boldsymbol{w} = \nabla P - \frac{1}{2}\boldsymbol{w}^2 \nabla\gamma + (\boldsymbol{w} \cdot \nabla\gamma)\boldsymbol{w}, \qquad \nabla \cdot \boldsymbol{w} = \boldsymbol{0} \qquad \text{in } \ \Omega, \tag{46}$$

with

$$P = 0, \qquad \gamma = 1 - \frac{1}{f^2} \tag{47}$$

Finally, we remark that (43) is well defined along the toroidal axis $re^{-\epsilon y} \to r_0, z \to 0$ provided that f exists in this limit (this is the case of $f^2 = \exp\{\Psi_\epsilon\}$ considered above).

[29] N. Sato, Z. Qu, D. Pfefferlé, and R. L. Dewar, Phys. Plasmas **28**, 112507 (2021). [30] H. Grad, The guiding center plasma, Proc. Symp. Appl. Math. 18, pp. 162-248 (1967).

Other Examples

It is not difficult to generalize the construction leading to (43) to obtain new solutions of both eq. (1) and anisotropic MHD. It is sufficient to replace $\sigma = x$ with a new harmonic function and solve for Ψ_ϵ. For example, taking $\epsilon, \epsilon' \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}$ one obtains

$$\sigma = e^{mx}\cos(my)\,,$$

$$\xi_\epsilon = \nabla[\varphi + \epsilon e^{mx}\cos(my)],$$

$$\Psi_\epsilon = \frac{1}{2}\left[\left(re^{-\epsilon e^{mx}\sin(my)} - r_0\right)^2 + z^2 e^{-2\epsilon' Z}\right], \tag{48}$$

$$w = f(\Psi_\epsilon)\xi_\epsilon.$$

- Relation with harmonic conjugate functions in two dimensions.
- The solution above breaks both continuous and discrete Euclidean symmetries.

Considerations on Steady Euler Flows and MHD Equilibria: Clebsch Reduction

Applying the Clebsch representation $\boldsymbol{w} = \nabla\Psi \times \nabla\Theta$ to eq. (2) gives

$$[\nabla \times (\nabla\Psi \times \nabla\Theta)] \times (\nabla\Psi \times \nabla\Theta) = \nabla\Psi \quad \text{in } \Omega, \tag{49}$$

which is equivalent to

$$\nabla \cdot [\nabla\Theta \times (\nabla\Psi \times \nabla\Theta)] = -1, \quad \nabla \cdot [\nabla\Psi \times (\nabla\Theta \times \nabla\Psi)] = 0, \quad \text{in } \Omega, \quad \Psi = \text{constant} \quad \text{on } \partial\Omega \tag{50}$$

- While in the study of eq. (1) the function Ψ was given, it is convenient to regard system (50) as coupled partial differential equations for the unknowns Ψ and Θ with Ω given.
- One expects that fixing Ψ will prevent, in general, the existence of regular solutions Θ fulfilling both equations in (50).
- Notice that boundary conditions on Ψ have been imposed to ensure that $\boldsymbol{w} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.

Considerations on Steady Euler Flows and MHD Equilibria: Variational Principle

Eq. (50) admits a variational principle. The target functional is

$$E'_\Omega = \frac{1}{2}\int_\Omega \left(|\nabla\Psi \times \nabla\Theta|^2 - \Psi\right) dV\,. \tag{51}$$

Assuming $\delta\Theta = \delta\Psi$ on $\partial\Omega$,

$$\delta E'_\Omega = -\int_\Omega \delta\Psi\,\{1 + \nabla\cdot[\nabla\Theta \times (\nabla\Psi \times \nabla\Theta)]\}\,dV - \int_\Omega \delta\Theta\,\nabla\cdot[\nabla\Psi \times (\nabla\Theta \times \nabla\Psi)]\,dV\,. \tag{52}$$

Hence, stationary points of the functional E'_Ω assign solutions of (50). Now suppose that solutions (Ψ, Θ) of (50) are sought in the Sobolev space $H^1(\Omega)$ with norm $||\cdot||_{H^1(\Omega)}$. From (51) it is clear that the functional E'_Ω is not coercive, i.e. it does not satisfy a condition of the form $E'_\Omega \geq c_1\,||\Psi||^2_{H^1(\Omega)} + c_2||\Theta||^2_{H^1(\Omega)} + C$ for some constants with $c1, c2, C \in R$, $c_1 > 0$, and $c_2 > 0$. Indeed, the value of (51) can be kept finite, $|E'_\Omega| < \infty$, even if $||\Psi||^2_{H^1(\Omega)}$, $||\Theta||^2_{H^1(\Omega)} \to \infty$ by setting $\Theta = \Psi$ while taking $||\nabla\Psi||^2_{L^2(\Omega)} = ||\nabla\Theta||^2_{L^2(\Omega)} \to \infty$ where $||\cdot||_{L^2(\Omega)}$ denotes the standard $L^2(\Omega)$ norm. The lack of coercivity prevents the application of variational methods [31] to establish the existence of a relative minimizer of E'_Ω, and thus a solution of (50) in the relevant function space.

[31] M. Struwe, The direct methods in the calculus of variations, in Variational Methods, 4th ed., Springer, p.4 (2010).

Considerations on Steady Euler Flows and MHD Equilibria: Variational Principle

The situation is different if the variable Ψ is fixed, i.e. if one considers eq. (23) arising from the functional E_Ω in the context of eq. (1). Using the change of variables $\Theta = \mu + \rho$ and coordinates $(x^1, x^2, x^3) = (\mu, \nu, \Psi)$,

$$E_\Omega = \frac{1}{2}\int_U d\Psi \int_D \left[\sum_{i,j=1}^{2} Ja_{ij}\rho_i\rho_j - 2\rho\frac{\partial(Ja_{\mu i})}{\partial x^i} + Ja_{\mu\mu}\right] d\mu d\nu\,, \tag{53}$$

For each $\Psi \in U$ we may therefore identify an energy functional

$$E_D = \frac{1}{2}\int_D \left[\sum_{i,j=1}^{2} Ja_{ij}\rho_i\rho_j - 2\rho\frac{\partial(Ja_{\mu i})}{\partial x^i} + Ja_{\mu\mu}\right] d\mu d\nu \geq \lambda\|\nabla_{(\mu,\nu)}\rho\|_{L^2(D)}^2 - 2c\|\rho\|_{L^2(D)}^2 - C. \tag{54}$$

Here, λ, c, and C are positive real constants and we used the strict ellipticity of Ja_{ij}. Recalling that $\langle\rho\rangle = 0$ and applying the Poincaré inequality,

$$E_D \geq \frac{\lambda}{4}\|\rho\|_{H^1(D)}^2 - \frac{4c^2}{\lambda} - C. \tag{55}$$

Hence, E_D is coercive with respect to the $H^1(D)$ norm since $E_D \to \infty$ when $||\rho||_{H^1(D)} \to \infty$. Then, for each Ψ there exist a relative minimizer $\rho \in H^1_{\text{per}}(D)$ of E_D, which corresponds to a solution of equation (30).

Considerations on Quasisymmetry

Quasisymmetry is a desirable feature for the confining magnetic field in stellarators, because it ensures steady confinement of the burning plasma within a finite volume of space [12]. A solution $\boldsymbol{w}$ of eq. (2) is quasisymmetric if there exists a quasisymmetry $\boldsymbol{u}(\boldsymbol{x})$ and a function $g(\Psi)$ with $\nabla g \neq \mathbf{0}$ such that

$$\boldsymbol{u} \times \boldsymbol{w} = \nabla g(\Psi), \qquad \boldsymbol{u} \cdot \nabla \boldsymbol{w}^2 = 0, \qquad \nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } \Omega. \tag{56}$$

Proposition 2: *suppose that $\boldsymbol{\xi} \in L^2_H(\Omega)$ is a harmonic vector field in a toroidal domain Ω foliated by nested toroidal surfaces corresponding to contours of a function $\Psi \in C^1(\overline{\Omega})$. Further assume that*

$$\boldsymbol{\xi} \cdot \nabla\Psi = 0 \qquad \text{in } \Omega,$$

and that $|\boldsymbol{\xi}|^2 = |\boldsymbol{\xi}|^2(\Psi)$. Then, the vector field $\boldsymbol{w} = f(\Psi)\boldsymbol{\xi} \in H^1_{\sigma\sigma}(\Omega)$, with $f \in C^1(\overline{\Omega})$, solves (2) and is quasisymmetric with quasisymmetry

$$\boldsymbol{u} = \boldsymbol{\xi} \times \nabla\Psi \in L^2_\sigma(\Omega).$$

The requirement $|\boldsymbol{\xi}|^2 = |\boldsymbol{\xi}|^2(\Psi)$ is a stringent condition related to the notion of isodynamic magnetic field [32]. Therefore, the existence of such configurations is nontrivial.

[12] E. Rodriguez, P. Helander, and A. Bhattacharjee, Phys. Plasmas **27**, 062501 (2020). [32] M. P. Bernardin, R. W. Moses, and J. A. Tataronis, The Physics of Fluids **29**, 2605 (1986).

Concluding Remarks

- We studied eq. (1), which determines a solenoidal vector field $\boldsymbol{w}$ such that both $\boldsymbol{w}$ and $\nabla \times \boldsymbol{w}$ are foliated by a family of nested toroidal surfaces Ψ.
- Eq. (1) represents a generalization of a eq. (2) encountered in fluid mechanics and MHD, which describes steady Euler flows and equilibrium magnetic fields.
- A general theory on the existence of solutions of eq. (2) is not available due its nontrivial characteristic surfaces. Analysis of the simpler problem posed by eq. (1) may therefore provide insight into the space of solutions of eq. (2).
- Theorem 1 shows that nontrivial solutions $\boldsymbol{w}$ in the class $C^{\infty}(\Omega)$ of eq. (1), where Ω is a hollow toroidal volume, always exist for a given family of smooth nested toroidal surfaces Ψ. The proof relies on the reduction of eq. (1) to a two-dimensional linear elliptic 2nd order PDE (30) for each toroidal surface with the aid of Clebsch potentials. Regular periodic solutions for these equations exist by elliptic theory and determine a smooth solution of problem (1).

Concluding Remarks

- Examples of smooth solutions in toroidal volumes were constructed analytically such that both the bounding surface and the solution are not invariant under continuous Euclidean isometries. Such solutions can be regarded as solutions of anisotropic MHD.
- The formulation of eq. (2) in terms of Clebsch potentials suggests that simultaneous optimization of the Clebsch potentials Ψ and Θ is needed to find solutions: the shape of the toroidal surfaces Ψ (and possibly the profile of the domain Ω itself) should be adjusted together with the variable Θ to accommodate the solution within Ω.
- If solutions are sought in the form $\boldsymbol{w} = f(\Psi)\boldsymbol{\xi}$, with $\boldsymbol{\xi}$ a harmonic vector field in Ω, solving (2) amounts to finding a harmonic vector field foliated by toroidal surfaces Ψ and such that the modulus $|\boldsymbol{\xi}|^2$ is itself a function of Ψ.
- If such kind of solutions of (2) could be found, they would also guarantee quasisymmetry, and thus magnetic confinement of a plasma within a finite volume of space.

Thank you for your attention!