Meta-Hierarchy Dynamics Unit
Research Report [2023/4~2024/3]

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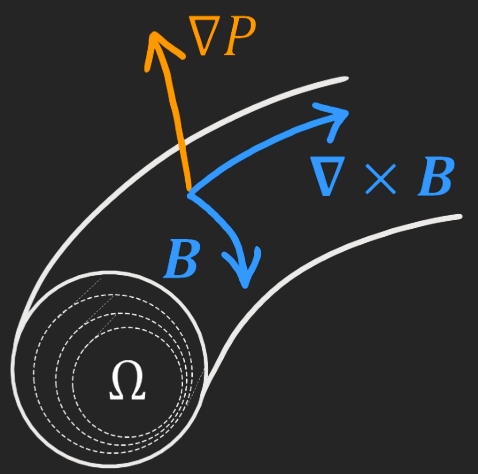
Magnetohydrostatics (MHS)

MHD Equilibria are described by the MHS equations

$$(\nabla\times\boldsymbol{B})\times\boldsymbol{B}=\mu_0\nabla P,\qquad \nabla\cdot\boldsymbol{B}=0\qquad \text{in }\ \Omega,\qquad \nabla P\times\boldsymbol{n}=\boldsymbol{0}\qquad \text{on }\ \partial\Omega. \tag{1}$$

Main Results

1. Given a set of nested flux surfaces $\Psi(\boldsymbol{x})$, there exists an integration factor $\lambda(\boldsymbol{x})$ and a solenoidal vector field $\boldsymbol{B}$ such that $(\nabla\times\boldsymbol{B})\times\boldsymbol{B}=\lambda\nabla\Psi$. Furthermore, this configurations can be interpreted as solutions of anisotropic MHS [1].
2. Dynamical system describing magnetic field turbulence around an MHS solution. This system can be used to rapidly construct MHD equilibria with nested flux surfaces [2].

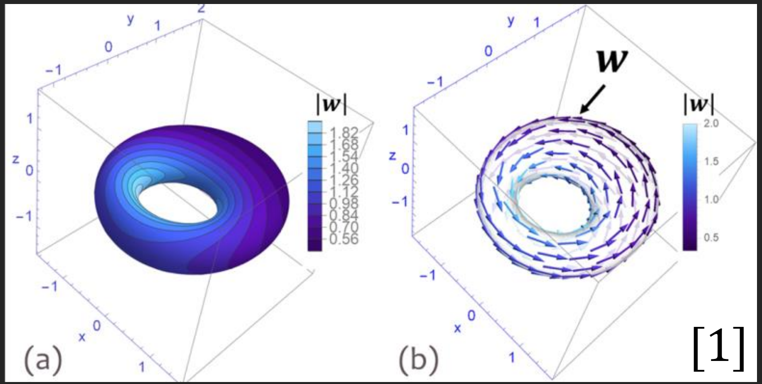


Papers

[1] N. Sato and M. Yamada, J. Math. Phys. **64**, 081505 (2023).
[2] N. Sato and M. Yamada, arxiv.org/abs/2311.03095.

Talks

[T1] N. Sato and M. Yamada, "A Reduced Ideal MHD System for Nonlinear Magnetic Field Turbulence in Plasmas with Approximate Flux Surfaces", The Grad conjecture in Fluid Mechanics and MHD: Theory and Applications, I-ste Namba, Osaka, 28 March 2024 (INVITED).
[T2] N. Sato, "MHS (Magnetohydrostatics)", Simons Hidden Symmetries and Fusion Energy Collaboration Australian Retreat, 12 December 2023.
[T3] N. Sato and M. Yamada, "On the Grad conjecture in anisotropic MHD", AAPPS-DPP 2023, F-5-I3, 15/11/2023 (INVITED).



Grad Conjecture Workshop (23-26 March 2024, I-Site Namba, Osaka)

Aim & Scope

Magnetohydrostatics (MHS) and the Euler equations are gathering momentum due to their relevance for practical applications, including the design of novel fusion reactors. The workshop mainly consisted of talks pertaining to MHS and the Euler equations, with both mathematicians and plasma physicists joining the event. One of the purposes of this gathering was to discuss the Grad conjecture from a broad perspective, why it is difficult to solve it, what it means to solve it, and envisage applications and significance for math and physics.

Grant

流体力学及び電磁流体力学における Grad 予想の数理と応用, 大阪公立大学数学研究所 文部科学省 共同利用・共同研究拠点 「数学・理論物理の協働・共創による 新たな国際的研究・教育拠点」 2023年 - 2024年, 佐藤直木（研究代表者）, 阿部健, 横山知郎, 吉田善章, 山田道夫

External links

- Workshop webpage (slides, video recordings, OCAMI report): https://sites.google.com/view/grad-conjecture/home?authuser=0
- OCAMI webpage: https://www.omu.ac.jp/orp/ocami-en/joint/

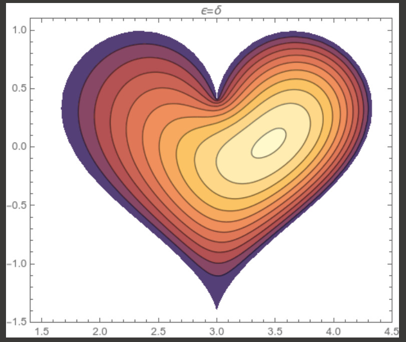
International Speakers

Federico Pasqualotto (UC Berkeley)
Javier Peñafiel-Tomás (ICMAT)
Daniel Peralta-Salas (ICMAT)
David Perrella (U. Western Australia)
David Pfefferlé (U. Western Australia)

National Speakers

Ken Abe (Osaka Metropolitan U.)
Yasuhide Fukumoto (Kyushu U.)
Yukimi Goto (Kyushu U.)
Kiori Obuse (Okayama U.)
Koji Ohkitani (Kyoto U.)
Naoki Sato (NIFS)
Ikkei Shimizu (Osaka U.)
Keiichirou Takeda (U. Tokyo)
Tomoo Yokoyama (Saitama U.)

Solution of the G-S eq. in a heart shaped domain by D. Pfefferlé



An Extended Hasegawa-Mima Equation For Nonlinear Drift Wave Turbulence in General Magnetic Configurations

We proposed [3-4] a generalization of the HM equation which accounts for drift wave turbulence in systems with strong magnetic field and electron density gradients $L\nabla\log B\sim L\nabla\log A_e\sim 1$:

$$\frac{\partial}{\partial t}\left[\lambda A_e\chi-\sigma\nabla\cdot\left(\frac{A_e\nabla_\perp\chi}{B^2}\right)\right]=\nabla\cdot\left[A_e\left(\sigma\frac{\boldsymbol{B}\cdot\nabla\times\boldsymbol{v}_{\boldsymbol{E}}^{\chi}}{B^2}-1\right)\boldsymbol{v}_{\boldsymbol{E}}^{\chi}\right]. \tag{2}$$

- $\boldsymbol{x}\in\Omega\subseteq\mathbb{R}^3, t\in[0,\infty)$
- $\chi(\boldsymbol{x},t)=\varphi(\boldsymbol{x},t)+\frac{\sigma}{2}\boldsymbol{v}_{\boldsymbol{E}}^2(\boldsymbol{x},t)$ (charged particle energy)
- $\boldsymbol{B}=\boldsymbol{B}(\boldsymbol{x})\neq\boldsymbol{0}$ (static magnetic field of arbitrary geometry)
- $A_e=A_e(\boldsymbol{x})$ (leading order electron spatial density, $n_e=A_e\exp(e\varphi/k_BT_e)$)
- $\nabla_\perp=-B^{-2}\boldsymbol{B}\times(\boldsymbol{B}\times\nabla)$
- $\boldsymbol{v}_{\boldsymbol{E}}^{\chi}=\boldsymbol{B}\times\nabla\chi/B^2$
- $\boldsymbol{v}_{\boldsymbol{E}}=\boldsymbol{B}\times\nabla\varphi/B^2$

HM eq. when
$\boldsymbol{B}=B_0\nabla z$,
$\log A_e=\log A_{e0}+\beta x$,
$\beta L\sim\epsilon$

Papers
[3] N. Sato and M. Yamada, Physica D: Nonlinear Phenomena **459**, 134031 (2024).
[4] 佐藤直木, 山田道夫, プラズマ核融合学会誌 **99** (5) 183-186 2023年5月.
Talks
[T4] N. Sato and M. Yamada, "Guiding Center Derivation, Hamiltonian Structure, and Nonlinear Stability of Steady Solutions of the Generalized Hasegawa-Mima Equation for Drift Wave Turbulence in Curved Magnetic Fields", 16aA307-9, JPS 2023 fall meeting, Tohoku University, Sendai, 16 September 2023.

A Collision Operator for Describing Dissipation in Noncanonical Phase Space

Suppose that the distribution function $f(\boldsymbol{z},t)$ with phase space coordinate $\boldsymbol{z}$ evolves in time t according to

$$\frac{\partial f}{\partial t} = -\{f,\mathfrak{H}\}_* + \left(\frac{\partial f}{\partial t}\right)_{\rm coll}, \tag{3}$$

where $\mathfrak{H}$ is the Hamiltonian (total energy), $\{\cdot,\cdot\}_*$ a noncanonical Poisson bracket, and $(\partial f/\partial t)_{\rm coll}$ a collision term.

Problem. What is the expression of $(\partial f/\partial t)_{\rm coll}$ for binary collisions that occur over short times scales and small spatial scales compared to the ideal dynamics described by the noncanonical Poisson bracket?

Theorem. *The collision term* $(\partial f/\partial t)_{\rm coll}$ *is given by* [5]

$$\left(\frac{\partial f}{\partial t}\right)_{\rm coll} = \mathcal{C}(f,f) = \frac{\partial}{\partial \boldsymbol{z}}\cdot\left[f\mathcal{J}\int f'\Pi\left(\mathcal{J}'\frac{\partial \log f'}{\partial \boldsymbol{z}'} - \mathcal{J}\frac{\partial \log f}{\partial \boldsymbol{z}}\right)d\boldsymbol{z}'\right]. \tag{4}$$

Here, Π is a symmetric covariant 2-tensor (interaction tensor) depending on the type of binary interactions, $\mathcal{J}$ the Poisson 2-tensor associated with the Poisson bracket, and the notation $f' = f(\boldsymbol{z}',t)$ and $\mathcal{J}' = \mathcal{J}(\boldsymbol{z}')$ was used.

Papers
[5] N. Sato and P. J. Morrison, arXiv:2401.15086 (2024).
Talks
[T5] N. Sato and P. J. Morrison, "On the Landau Collision Operator in Noncanonical Phase Space and the resulting modification of the moment equations governing fluid dynamics", Fluids in Seoul 2024, June E Huh Center for Mathematical Challenges (HCMC), 18 January 2024 (INVITED).

Current & Future Projects

- Enhance International Collaborations
- Interact with the math, fluid mechanics, & theoretical physics communities

Current projects

MHS without incompressibility > asymmetric solutions probably exist
Enstrophy conservation in relativistic flows (with K. Takeda)
Numerical simulation of Navier-Stokes equations with Clebsch potentials [6] (with S. Murai and Z. Yoshida)
Statistical mechanics of pair plasmas [7]

International collaborations under development

N. Duignan (U. Sydney) > Differential geometric formulation of Nambu dynamics > some new results [8]
D. Pfefferlé (UWA), D. Perrella (UWA), and F. Pasqualotto (UC Berkeley) > MHS existence & regularity
H. Abdelhamid (Mansoura U.) > Dynamo theory in XMHD
M. Materassi (CNR) > Metriplectic formalism with stochastic PDEs and fractional diffusion

Papers

[6] S. Murai, N. Sato, and Z. Yoshida, doi.org/10.48550/arXiv.2305.16673.
[7] N. Sato, Physics of Plasmas **30**, 042503 (2023).
[8] N. Sato, Progress of Theoretical and Experimental Physics **2024**, Issue 4, 049201 (2024).